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A SIMPLE MESHLESS ALGORITHM FOR STEADY GROUNDWATER FLOW GOVERNED BY LAPLACE EQUATION

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Abstract

The boundary knot method (BKM) based on the nonsingular general solutions, is a kind of typical boundary-type meshfree collocation technique. Since the nonsingular general solution takes the place of the singular fundamental solution to obtain the numerical solution, only boundary points are needed during the whole solution procedures. This paper concerns a numerical investigation of the steady fluid flow problems, which is described by boundary value problems. Our purpose is to apply the BKM to deal with Laplace-type problems from a new perspective. Numerical results show that the BKM can be successfully applied to simulate steady fluid flow problems with two and three dimensions.

Introduction

In recent years, a variety of boundary-type meshfree methods [1, 2] have been introduced to various engineering problems in order to avoid the tedious mesh generation. For example, the behavior of a fluid flow system can be represented by the Laplace equation with boundary conditions. A number of pioneers have adopted traditional grid-based methods to simulate fluid flow problems governed by Laplace equations [3, 4, 5, 6]. For meshless methods, Li and his co-workers [7] applied collocation method with radial basis functions to simulate groundwater contaminant transport. Long and his coworkers [8] used the Multiquadrics method to obtain the numerical solutions of Laplace equation of 2D steady groundwater flow, which was also extended to the research of Zhou and Zhang [9]. G. H. Schmitz and J. Edenhofer [10] investigated the exact closed-form solution of the two-dimensional Laplace equation for steady groundwater flow with non-linearized free-surface boundary conditions. The method of fundamental solutions (MFS) has been successfully applied to groundwater flow problems governed by Laplace equations [11]. Compared with the traditional numerical methods [12], the MFS implementation does not require integrations and discretization of the physical boundary as well as the physical domain.

As is known to all, the fictitious boundary of the MFS is somewhat arbitrary and not trivial to determine. In this case, the boundary knot method (BKM) [13, 14, 15] was proposed to alleviate the shortage of the MFS, using the non-singular general solutions instead of the singular fundamental solutions. The BKM has the advantages that the collocation and source points can be placed on the same physical boundary of the solution domain.

From what has been described above, we use the BKM to investigate the steady fluid flow problems governed by Laplace equations, which are approximated by the Helmholtz equation with small wavenumbers λ. Numerical results are compared with analytical solutions to show the applicability of the BKM. In order to simplify calculating, this paper only concerns the Dirichlet boundary. Section 2 briefly introduces the BKM formulation for the Laplace-type problems. Followed by Section 3, two numerical examples are presented and then we calculate their relative errors. Some conclusions are made in Section 4.

The main procedure of the BKM

To illustrate our theory, we consider the following governing equations for the two-dimensional steady fluid flow:

$$\upsilon^x(x,y) = -K\frac{\partial u(x,y)}{\partial x},\quad (x,y)\in\Omega \tag{1}$$

$$\upsilon^y(x,y) = -K\frac{\partial u(x,y)}{\partial y},\quad (x,y)\in\Omega \tag{2}$$

$$\frac{\partial \upsilon^x}{\partial x} + \frac{\partial \upsilon^y}{\partial y} = f(x,y),\quad (x,y)\in\Omega \tag{3}$$

where u, f are hydraulic head and source or sink, respectively. K is hydraulic conductivity. Ω means the physical domain in $\boldsymbol{R}^d$, d is the dimensionality. The head and fluxes boundary conditions are given by

$$u = \bar{u},\quad (x,y)\in\Gamma_D \tag{4}$$

$$\frac{\partial u}{\partial n} = \bar{q},\quad (x,y)\in\Gamma_N \tag{5}$$

where $\bar{u},\bar{q}$ are given hydraulic head on Dirichlet boundary Γ_D and source or sink on Neumann boundary Γ_N, respectively. $\Gamma_D\cup\Gamma_N = \partial\Omega$ $(\Gamma_D\cap\Gamma_N \neq \emptyset)$ is the boundary of flow

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field Ω. In this study, we only concern that there is no source or sink term, i.e. $f(x, y) = 0$. Eqs. (1)-(3) can be combined as a special case of the Poisson equation, i.e. the Laplace equation. It is well-known that the fundamental solution of the Laplace equation has singularity at the origin, but there is no non-singular general solution for the Laplace equation. Chen et al. [16] used the high-order general solutions of the Helmholtz and modified Helmholtz equations to evaluate the particular solution. However, a new parameter is introduced during the whole solution process which is somewhat arbitrary. Under such circumstance, we seek to find a new way to simulate the Laplace-type problems.

In this study, we have the non-singular general solution for the following Helmholtz equation:

$$\nabla^2 u + \lambda^2 = 0, \tag{6}$$

where $\nabla^2 = \Delta$ is a Laplace operator, λ is the wavenumber. Our fundamental theory can be stated as below. Since the numerical solutions are approximation of the exact solutions, we have the idea that the non-singular general solution for Helmholtz equations is a good approximation to the Laplace equation:

$$\Delta u = 0. \tag{7}$$

This is under the assumption that the wavenumber λ is sufficiently small. The non-singular general solution of Eq. (6) is given by $u^*(r) = J_0(r)$, where J denotes the Bessel function of the first kind, r represents the Euclidean norm distance. Since there is no singularity in the non-singular general solution $J_0(r)$, all collocation knots are placed on the physical boundary and can be used as either source or collocation points. Using the non-singular general solution $J_0(r)$, the solution of Eq. (6) can be approximated by

$$u(x_i, y_i) = \sum_{j=1}^{N} \alpha_j u^*(r_{ij}), \tag{8}$$

where j is the index of source points on physical boundary, N denotes the total number of boundary knots, α_j are the unknown coefficients and $r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ the Euclidean distance, where $\boldsymbol{i}$ stands for index of collocation points on physical boundary. By collocating boundary Equations (4) and (5), we have:

$$\sum_{j=1}^{N} \alpha_j u^*(r_{ij}) = \bar{u}(x_i, y_i), (x_i, y_i) \in \Gamma_D, \tag{9}$$

$$\sum_{j=1}^{N} \alpha_j \frac{\partial u^*(r_{kj})}{\partial n} = \bar{q}(x_k, y_k), (x_k, y_k) \in \Gamma_N. \tag{10}$$

Equations (9)-(10) can be written in the following linear matrix system:

$$A\alpha = b, \tag{11}$$

where $A = (A_{ij})$ is an interpolation matrix and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ are the unknown coefficients. It should be noticed that, due to the global interpolation approach, the BKM produces a highly ill-conditioned and dense matrix system when a large number of boundary knots are used [19, 20]. This problem needs further investigations.

Numerical example and discussions

To examine the accuracy and stability of the proposed method given in the above sections, we test two benchmark cases of homogeneous Laplace-type problems. The relative average errors (root mean-square relative error: RMSE) are:

$$\text{RMSE} = \sqrt{\frac{1}{N_t} \sum_{j=1}^{N_t} \left| \frac{u(X_j) - \tilde{u}(X_j)}{u(X_j)} \right|^2},$$

for $|u(X_j)| \geq 10^{-3}$, or

$$\text{RMSE} = \sqrt{\frac{1}{N_t} \sum_{j=1}^{N_t} \left| u(X_j) - \tilde{u}(X_j) \right|^2},$$

for $|u(X_j)| < 10^{-3}$, where j is the index of test points, $u(X_j)$ and $\tilde{u}(X_j)$ are the exact and numerical solutions on the test point X_j, respectively. N_t means the total number of test points.

Circular domain

The following exact solution will be compared with our numerical solutions

$$u(x, y) = x + y. \tag{12}$$

A unit circular domain $\Omega = \{(x, y) | x^2 + y^2 \leq 1\}$ is taken into account with Dirichlet boundary condition only.

Table 1. Relationship among wavenumber λ, condition number Cond and relative average error RMSE, using boundary point number N=20

λ	Cond	RMSE
0.1	8.4037×10^{18}	9.4400×10^{-4}
0.01	4.2365×10^{18}	9.5106×10^{-6}
0.001	1.2753×10^{18}	1.5500×10^{-2}
0.0001	8.7645×10^{18}	3.3791×10^{-7}

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To investigate the influence of the wavenumber λ on this problem, we fix the boundary point number N = 20. After calculating with MATLAB, relative average errors and condition numbers of the interpolation matrix are given in Table 1. We find that the relative average errors are acceptable for various wavenumbers. In particular, the optimum average relative error RMSE=3.3791×10^{-7} is obtained when the wavenumber λ= 0.0001. Meanwhile, the condition numbers of the interpolation matrix are relatively small, which contributes to the stability of the numerical solutions.

Table 2. Relationship among boundary point number, condition number Cond and relative average error RMSE, fixing wavenumber λ =0.0001

N	Cond	RMSE
10	5.6106×10^{17}	4.7123×10^{-7}
50	3.0560×10^{19}	3.4738×10^{-7}
100	2.3903×10^{19}	4.8530×10^{-7}
150	1.0488×10^{20}	3.0738×10^{-7}

The wavenumber λ=0.0001 is fixed, while the influence of the boundary point numbers on relative average errors and condition numbers of the interpolation matrix are given in Table 2. We can see that the RMSEs are very small, at 3.0738×10^{-7}, when N=150. At the same time, the condition number of the interpolation matrix increases with the increasing boundary point numbers.

Annulus domain

Here, we choose the following exact solution:

$$u(x,y) = \sin x \sinh y + \cos x \cosh y, \qquad (13)$$

which lies in the annulus domain $\Omega = \{(x,y): 0.5^2 < x^2 + y^2 < 1^2\}$ with the outer and the inner boundaries $\Gamma_{out} = \{(x,y): x^2+y^2 = 1^2\}$ and $\Gamma_{in} = \{(x,y): x^2+y^2 = 0.5^2\}$, respectively.

Table 3. Relationship among wavenumber λ, condition number Cond and relative average error RMSE, using boundary point number N=20

λ	Cond	RMSE
0.1	2.2374×10^{33}	0.0011
0.01	2.5997×10^{33}	0.0399
0.001	6.9172×10^{33}	0.0337

With fixing boundary point number N = 20, Table 3 shows the relative average errors and condition numbers of the interpolation matrix for different wavenumbers. It is clear that the relative average errors are also acceptable for various wavenumbers. When the wavenumber λ= 0.1, we can calculate good RMSE= 0.0011. Meanwhile, the condition numbers of the interpolation matrix are much larger than the previous case, which may lead to the instability of a method.

Table 4. Relationship among boundary point number, condition number Cond and relative average error RMSE, fixing wavenumber λ =0.1

N	Cond	RMSE
10	2.9685×10^{33}	0.0011
20	7.2374×10^{33}	0.0011
30	5.0558×10^{33}	0.0011
50	4.1852×10^{19}	0.0011

Similarly, it is clear from Table 4 that the RMSE maintains stable at 0.0011, because of the extremely large condition numbers of the interpolation matrix, at about 10^{33}.

Conclusions

The BKM is applied to steady-state fluid flow problems in this paper and this method avoids the mesh-generation. Numerical examples of steady fluid flow problems show the feasibility of the BKM, with acceptable results. Since the BKM only needs boundary data of the physical domain during the whole problem, it is promising to deal with more complicated fluid flow problems by choosing the optimal location and number of collocation points.

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References

[1] J. T. Chen, Y. T. Lee, S. R. Yu, and S. C. Shieh, "Equivalence between the Trefftz method and the method of fundamental solution for the annular Green's function using the addition theorem and image concept," Engineering Analysis with Boundary Elements, Vol. 33, No. 5, pp. 678-688, 2009.

ISSN:2319-7900

[2] Y. Gu, W. Chen, and C. Z. Zhang, "A meshless singular boundary method for three-dimensional elasticity problems," International Journal for Numerical Methods in Engineering, Vol. 107, pp. 109-126, 2016.
[3] N. Z. Sun, Mathematical Modeling of Groundwater Pollution. Springer New York, 1996.
[4] E. J. Diersch, FEFLOW: finite element modeling of flow, mass and heat transport in porous and fractured media. Springer Berlin Heidelberg, 2014.
[5] H. Ahmadi, and M. Manteghian, "A new algorithm that developed finite difference method for Solving Laplace equation for a plate with four different constant temperature boundary conditions," Research Journal of Applied Sciences Engineering & Technology, Vol. 4, No. 22, pp. 4630-4635, 2012.
[6] T. Takahashi, and T. Matsumoto, "An application of fast multipole method to isogeometric boundary element method for Laplace equation in two dimensions," Engineering Analysis with Boundary Elements, Vol. 36, No. 12, pp. 1766-1775, 2012.
[7] J. Li, Y. Chen, and D. Pepper, "Radial basis function method for 1-D and 2-D groundwater contaminant transport modeling," Computational Mechanics, Vol. 2, No. 1-2, pp. 10-15, 2003.
[8] Y. Q. Long, W. Li, Y. G. Li, and Z. P. Yang, "Application of Multi-Quadric method for numerical simulation of steady groundwater flow," Journal of Hydrolic Engineering, Vol. 42, No. 5, pp. 572-578, 2011. [in Chinese]
[9] D. L. Zhou, and J. Zhang, "The Multiquadric method for steady flow of groundwater," Journal of Liaoning Normal University (Natural Science Edition), Vol. 27, No. 4, pp. 399-402, 2004. [in Chinese]
[10] G. H. Schmitz, and J. Edenhofer, "Exact closed-form solution of the two-dimensional Laplace equation for steady groundwater flow with non-linearized free-surface boundary condition," Water Resources Research, Vol. 36, No. 7, pp. 1975-1980, 2000.
[11] F. Z. Wang, and K. H. Zheng, "The method of fundamental solutions for steady-state groundwater flow problems," Journal of the Chinese Institute of Engineers, Vol. 39, No. 2, pp. 236-242, 2016.
[12] J. T. Chen, C. Y. Yueh, Y. L. Chang, and C.C. Wen, "Why dual boundary element method is necessary," Engineering Analysis with Boundary Elements, Vol. 76, pp. 59-68, 2017.
[13] A. Canelas, and B. Sensale, "A boundary knot method for harmonic elastic and viscoelastic problems using single-domain approach," Engineering Analysis with Boundary Elements, Vol. 34, pp. 845-855, 2010.
[14] F. Z. Wang, W. Chen, and X. R. Jiang, "Investigation of regularization techniques for boundary knot method," International Journal for Numerical Methods in Engineering, Vol. 26, pp. 1868-1877, 2010.
[15] M. Dehghan, and R. Salehi, "A boundary-only meshless method for numerical solution of the Eikonal equation," Computational Mechanics, Vol. 47, pp. 283-294, 2011.
[16] W. Chen, L. J. Shen, Z. J. Shen, and G. W. Yuan, "Boundary knot method for Poisson equations," Engineering Analysis with Boundary Elements, Vol. 29, pp. 756-760, 2005.
[17] E. Kita, N. Kamiya, and Y. Ikeda, "An application of trefiz method to the sensitivity analysis of two-dimensional potential problem," International Journal for Numerical Methods in Engineering, Vol. 38, No. 13, pp. 2209-2224, 1995.
[18] F. Z. Wang, W. Chen, and L. Ling, "Combinations of the method of fundamental solutions for general inverse source identification problems," Applied Mathematics and Computation, Vol. 219, No. 3, pp. 1173-1182, 2012.
[19] F. Z. Wang, and K. H. Zheng, "Analysis of the boundary knot method for 3D Helmholtz-type problems," Mathematical Problems in Engineering, Volume 2014, Article ID 853252, pp. 1–9, 2014.
[20] F. Z. Wang, and Z. X. Ma, "Meshless collocation method for inverse source identification problems," Advances in Applied Mathematics and Mechanics, Vol. 7, No. 4, pp. 496-509, 2015.

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