International Journal of Advanced Computer Technology (IJACT)

ISSN:2319-7900

ON RELIABILITY EVALUATION OF A PROBABILISTIC NETWORK UNDER TIME AND COST CONSTRAINTS

Jsen-Shung Lin, Department of Information Management, Central Police University, Taoyuan, Taiwan 333, R.O.C

Abstract

Many real-world systems such as communication systems, transportation systems, and logistics/distribution systems that play important roles in our modern society can be regarded as probabilistic networks whose transmission time and transmission cost are independent, finite and multi-valued random variables. Such a network is indeed a multistate system with multistate components and so its reliability for level (d,c), i.e., the probability that the shortest transmission time from a specified source node to another specified sink node is less than or equal to d and the total transmission cost is no more than c, can be computed in terms of minimal cut vectors to level (d,c) (named (d,c)-MCs here). The main objective of this paper is to present a simple algorithm to search for all (d,c)-MCs of such a network and then to calculate its reliability in terms of such (d,c)-MCs by further applying the state-space decomposition method. A numerical example is given to illustrate the proposed method.

Introduction

System reliability is an important indicator in the planning, designing, and operation of a real-world system. Traditionally, it is assumed that the system under study is represented by a probabilistic graph in a binary-state model, and the system operates successfully if there exists one or more paths from the source node to the sink node. In such a case, reliability is considered as a matter of connectivity only and so it does not seem to be reasonable as a model for some real-world systems. Many physical systems such as communication systems, transportation systems and logistics/distribution systems can be regarded as probabilistic networks whose transmission time and transmission cost are independent, limited, and integer-valued random variables. For such a network, it is very practical and desirable to evaluate its reliability for level (d,c), i.e., the probability that the shortest transmission time from the source node to the sink node is less than or equal to d and the total transmission cost is no more than c.

In fact, reliability evaluation can be carried out in terms of minimal pathsets (MPs) or minimal cutsets (MCs) in the binary-state model case, and (d,c)-MCs (i.e., minimal cut vectors to level (d,c) [2], lower critical connection vector to level (d,c) [3], or upper boundary points of system level (d,c) [9]) for each level (d,c) in the multistate model case. The probabilistic network with random transmission times and transmission costs here can be treated as a multistate system of multistate components and so the need of an efficient algorithm to search for all of its (d,c)-MCs arises. The main purpose of this article is to present an intuitive algorithm to generate all (d,c)-MCs of such a network and then to compute its reliability in terms of such (d,c)-MCs by further applying the state space decomposition method [3].

Assumptions

A manufacturing system, transportation systems, and logistics/distribution system can be represented by a probabilistic network. Let $G=(N,A,L,U)$ be such a network with the unique source node s and the unique sink node t, where N is the set of nodes, $A=\{a_i \mid 1\le i\le n\}$ is the set of arcs, $L=(l_1,l_2,...,l_n)$ and $U=(u_1,u_2,...,u_n)$, where l_i and u_i denote the minimum, and maximum transmission time of each arc a_i, respectively. Such a probabilistic network is assumed to further satisfy the following assumptions:

1. Each node is perfectly reliable. Otherwise, the network will be enlarged by treating each of such nodes as an arc [1].
2. The transmission time and transmission cost of each arc a_i are integer-valued random variables that takes integer values according to a given distribution.
3. The transmission times and transmission costs of different arcs are statistically independent.

Assumption 3 is made just for convenience. If it fails in practice, the proposed algorithm to search for all (d,c)-MCs is still valid except that the reliability evaluation in terms of such (d,c)-MCs should take the joint probability distributions of all arcs into account.

Let $X=(X_1,X_2,...,X_n)$ be a system-state vector (i.e., the current transmission time of each arc a_i under X is x_i, where x_i takes integer values from l_i to u_i, and $V(X)$, the shortest transmission time from s to t under X. Such a function $V(X)$ plays the role of the so-called structure function of a multistate system with $V(L)=h$ and $V(U)=k$. Under

ISSN:2319-7900

the system-state vector $X = (X_1, X_2, ..., X_n)$, the arc set A has the following three important subsets: $N_X = \{a_i \in A \mid x_i < u_i\}$, $B_X = \{a_i \in A \mid x_i = u_i\}$ and $S_X = \{a_i \in A \mid V(X + e_i) > V(X)\}$ where $e_i = (\delta_{i1}, \delta_{i2}, ..., \delta_{in})$, with $\delta_{ij} = 1$ if $j = i$ and 0 if $j \neq i$. In fact, $A = S_X \cup (N_X \setminus S_X) \cup B_X$ is a disjoint union of A under X.

A system-state vector X is said to be a (d,c)-MC if and only if (1) its system level is d (i.e., $V(X) = d$), (2) each arc without maximum transmission time under X is sensitive (i.e., $N_X = S_X$), and (3) the total transmission cost is no more than c. If level (d,c) is given, then the probability that the shortest transmission time from the source node s to the sink node t is less than or equal to d and the total transmission cost is no more than c, is taken as the system reliability.

Model Building

Suppose that $P^1, P^2, ..., P^m$ are the collection of all MPs of the system. For each P^j, the transmission time from the source node s to the sink node t is defined as the sum of the transmission times of all arcs in it. Hence, $V(X) = \min_{1 \le j \le m}\{\sum_i \{x_i \mid a_i \in P^j\}\}$ is the shortest transmission time from s to t under X. Because $V(X)$ is non-decreasing in each argument under X, the probabilistic network with random transmission times and transmission costs can be viewed as a multistate monotone system with the structure function $V(\cdot)$ [2].

A necessary condition for a system vector X to be a (d,c)-MC is stated in the following theorems. Our algorithm relies mainly on such a result.

Theorem 1. If X is a (d,c)-MC, then $S_X \subseteq \cap_j \{P^j\}\sum_i \{x_i \mid a_i \in P^j\} = d\}$ and $\sum_{i=1}^n \text{cost}(i, x_i) \le c$.

Proof: Suppose on the contrary that there exists a MP P^r with $\sum_i \{x_i \mid a_i \in P^j\} = d$ such that $S_X \setminus P^r = \{a_i \mid a_i \in S_X \, and \, a_i \notin P^r\} \neq \phi$. Choose an $a_i \in S_X \setminus P^r$ and let $Y = X + e_i = (x_1, x_2, ..., x_{i-1}, x_i + 1, x_{i+1}, ..., x_n) = (y_1, y_2, ..., y_n)$. Then $\sum_i \{y_i \mid a_i \in P^r\} = \sum_i \{x_i \mid a_i \in P^r\} = d$ due to the fact that $a_i \notin P^r$ and so $V(Y) = d$ which contradicts the fact that $a_i \in S_X$. Hence, $S_X \subseteq \cap_j \{P^j\}\sum_i \{x_i \mid a_i \in P^j\} = d\}$ and $\sum_{i=1}^n \text{cost}(i, x_i) \le c$.

Theorem 2. If X is a (d,c)-MC, then there exists at least one MP $P^r = \{a_{r1}, a_{r2}, ..., a_{rn_r}\}$ such that the following conditions are satisfied:

$x_{r1} + x_{r2} + ... + x_{rn_r} = d$

$l_i \le x_i \le u_i$ for all $a_i \in P^r$

$x_i = u_i$ for all $a_i \notin P^r$

$\sum_{i=1}^n \text{cost}(i, x_i) \le c$

Proof: Let J be the non-empty index set of MPs such that $\sum_i \{x_i \mid a_i \in P^j\} = d$ for $j \in J$ and $\sum_i \{x_i \mid a_i \in P^j\} > d$ for $j \notin J$. Choose a P^r with $r \in J$, say $P^r = \{a_{r1}, a_{r2}, ..., a_{rn_r}\}$, then $\sum_i \{x_i \mid a_i \in P^j\} = d$, i.e.,

$x_{r1} + x_{r2} + ... + x_{rn_r} = d$

$l_i \le x_i \le u_i$ for all $a_i \in P^r$

By Theorem 1,

$A \setminus P^r \subseteq A \setminus \cap_{j \in J}\{P^j \mid \sum_i \{x_i \mid a_i \in P^j\} = d\} \subseteq A \setminus S_X = B_X$, i.e.,

$x_i = u_i$ for $a_i \notin P^r$. Finally, the total transmission cost is no more than c, i.e.,

$$\sum_{i=1}^n \text{cost}(i, x_i) = \text{cost}(1, x_1) + ... + \text{cost}(n, x_n) \le c.$$

Any vector $X = (X_1, X_2, ..., X_n)$ which satisfies constraints (1) - (4) simultaneously will be taken as a (d,c)-MC candidate. A (d,c)-MC is obviously a (d,c)-MC candidate by Theorem 2. By definition, a (d,c)-MC candidate X is a (d,c)-MC if (1) $V(X) = d$, (2) $N_X = S_X$, and (3) the total transmission cost is no more than c.

Theorem 3. If the network is parallel-series, then each (d,c)-MC candidate is a (d,c)-MC.

Proof: Such a network can be considered as the parallel of its MPs $P^1, P^2, ..., P^m$. Let X be a (d,c)-MC candidate which is generated with respect to P^r according to Lemma 2. Since the network is parallel-series, $P^j \cap P^r = \phi$ for each $j \neq r$. Then $\sum_i \{x_i \mid a_i \in P^r\} = d$ and $\sum_i \{x_i \mid a_i \in P^r\} = \sum_i \{u_i \mid a_i \in P^j\} \ge V(U) = k > d$ for each $j \neq r$. In particular, $V(X) = d$ and $N_X \subseteq \cap_j \{P^j \mid \sum_i x_i \mid a_i \in P^j\} = d\} = P^r$. Hence, X is a (d,c)-MC.

□

ISSN:2319-7900

Algorithm

Suppose that all MPs, $P^1, P^2, ..., P^m$, have been stipulated in advance [4, 7, 14-15], the family of all (d,c)-MCs can then be derived by the following steps:

Step 1. For each $P^r = \{a_{r1}, a_{r2}, ..., a_{rm_r}\}$, find all integer-valued solutions of the following constraints by applying an implicit enumeration method:

(1) $x_{r1} + x_{r2} + ... + x_{rm_r} = d$

(2) $l_i \le x_i \le u_i$ for all $a_i \in P^r$

(3) $x_i = u_i$ for all $a_i \notin P^r$

(4) $\sum_{i=1}^{n} \text{cost}(i, x_i) \le c$

Step 2. Check each candidate X one at a time whether it is a (d,c)-MC:

(a) If the network is parallel-series, then each candidate is a (d,c)-MC.

(b) If the network is non parallel-series, then check each candidate whether it is a (d,c)-MC as follows:

(2.1) If there exists an $j \ne r$ such that $\sum_i \{x_i \mid a_i \in P^j\} < d$, then X is a (d,c)-MC and go to step (2.4).

(2.2) Let index set $I = \{j \mid \sum_i \{x_i \mid a_i \in P^j\} = d\}$.

(2.3) If there exists an $a_i \in A \setminus \cap_{j \in I} P^j$ such that $x_i \ne u_i$, then X is not a (d,c)-MC, else X is a (d,c)-MC.

(2.4) Next candidate.

An Example

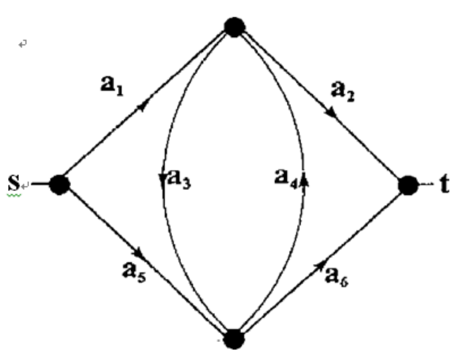

Figure 1: A bridge network.

Table 1. Probability distributions of transmission time and transmission cost

arc	time	cost	Probability
a_1	3	Cost(1,3)=3	0.30
	2	Cost(1,2)=4	0.50
	1	Cost(1,1)=5	0.20
a_2	2	Cost(2,2)=2	0.40
	1	Cost(2,1)=3	0.50
a_3,	2	Cost(3,2)=2	0.80
	1	Cost(3,1)=3	0.20
a_4	2	Cost(3,2)=2	0.80
	1	Cost(3,1)=3	0.20
a_5	2	Cost(4,2)=2	0.70
	1	Cost(4,1)=3	0.30
a_6	3	Cost(5,3)=3	0.60
	2	Cost(5,2)=4	0.35
	1	Cost(5,1)=5	0.05

It is known that $L = (l_1, l_2, l_3, l_4, l_5, l_6) = (1,1,1,1,1,1)$ with $V(L) = 2$, $U = (u_1, u_2, u_3, u_4, u_5, u_6) = (3,2,2,2,2,3)$ with $V(U) = 5$, and there exists four MPs; $P^1 = \{a_1, a_2\}, P^2 = \{a_1, a_3, a_6\}, P^3 = \{a_2, a_4, a_5\}, P^4 = \{a_5, a_6\}$.

Hence, $n = 6, m = 4$ and the system has 4 levels: 2, 3, 4, 5. Given $d = 4$ and $c = 16$, the family of (4,16)-MCs is derived as follows:

Step 1. For $P^1 = \{a_1, a_2\}$, find all integer-valued solutions of the following constraints by applying an implicit enumeration method:

$x_1 + x_2 = 4$

$1 \le x_1 \le 3$

$1 \le x_2 \le 2$

$x_3 = 2, x_4 = 2,$ and $x_5 = 3$.

Two feasible solutions are $(x_1, x_2, x_3, x_4, x_5, x_6) = (2,2,2,2,2,3)$ and $(x_1, x_2, x_3, x_4, x_5, x_6) = (3,1,2,2,2,3)$.

(1) When $(x_1, x_2, x_3, x_4, x_5, x_6) = (2,2,2,2,2,3)$, the total transmission cost is

$$\sum_{i=1}^{5} c(i, x_i) = 4 + 2 + 2 + 2 + 2 + 3 = 15 \le 16$$

(2) When $(x_1, x_2, x_3, x_4, x_5, x_6) = (3,1,2,2,2,3)$, the total transmission cost is

$$\sum_{i=1}^{5} c(i, x_i) = 3 + 3 + 2 + 2 + 2 + 3 = 15 \le 16$$

Two (4,16)-MC candidates $X^1 = (2,2,2,2,2,3)$ and $X^2 = (3,1,2,2,2,3)$ are obtained.

ISSN:2319-7900

Step 2. Check $X^1 = (2,2,2,2,2,3)$ whether it is a (4,16)-MC.

(2.1) $\sum_i \{x_i \mid a_i \in P^j\} > 4$ for each P^j with $j \neq 1$.

(2.2) $J = \{j \mid \sum_i \{x_i \mid a_i \in P^j\} = 4\} = \{1\}$.

(2.3) $X^1 = (2,2,2,2,2,3)$ is a (4,16)-MC.

(2.4) Next candidate (i.e., check $X^2 = (3,1,2,2,2,3)$ whether it is a (4,16)-MC.

(2.1) $\sum_i \{x_i \mid a_i \in P^j\} > 4$ for each P^j with $j \neq 1$..

(2.2) $J = \{j \mid \sum_i \{x_i \mid a_i \in P^j\} = 4\} = \{1\}$.

(2.3) $X^2 = (3,1,2,2,2,3)$ is a (4,16)-MC.

Step 1. For $P^2 = \{a_1, a_3, a_6\}$, find all integer-valued solutions of the following constraints by applying an implicit enumeration method:

$x_1 + x_3 + x_6 = 4$

$1 \le x_1 \le 3$

$1 \le x_3 \le 2$

$1 \le x_6 \le 3$

$x_2 = 2$, $x_4 = 2$, and $x_5 = 2$.

Three feasible solutions are $(x_1, x_2, x_3, x_4, x_5, x_6) = (1,2,1,2,2,2)$, $(x_1, x_2, x_3, x_4, x_5, x_6) = (1,2,2,2,2,1)$, and $(x_1, x_2, x_3, x_4, x_5, x_6) = (2,2,1,2,2,1)$.

………

The result is listed in Table 2.

Table2. List of all (4,16)-MCs

P^r	(4,16)-MC candidate	(4,16)-MC?
P^1	$X^1 = (2,2,2,2,2,3)$	Yes
	$X^2 = (3,1,2,2,2,3)$	Yes
P^2	$X^3 = (1,2,1,2,2,2)$	No
	$X^4 = (1,2,2,2,2,1)$	No
	$X^5 = (2,2,1,2,2,1)$	No
P^3	$X^6 = (3,2,2,1,1,3)$	No
	$X^7 = (3,1,2,2,1,3)$	No
	$X^8 = (3,1,2,1,2,3)$	No
P^4	$X^9 = (3,2,2,2,2,2)$	Yes
	$X^{10} = (2,2,2,2,1,3)$	Yes

Reliability Evaluation

If $Y^1, Y^2, \ldots, Y^{m_{(d,c)}}$ are the collection of all (d,c)-MCs, then the system reliability for level (d,c) is defined as $R_{(d,c)} = \Pr\{\cup_{i=1}^{m_{(d,c)}} \{X \mid X \le Y^i\}\}$. To compute it, several methods such as inclusion-exclusion [5, 8], disjoint subset [10], and state-space decomposition [3] are available. Here we apply the state-space decomposition method to the example and obtain that $R_{(4,16)} = \Pr\{\cup_{i=1}^{m_{(4,16)}} \{X \mid X \le Y^i\}\} = 0.9496$.

Conclusion

Given all MPs that are stipulated in advance, the proposed method can generate all (d,c)-MCs of a probabilistic transportation/logistics system whose transmission times and transmission costs are random variables for each level (d,c). The system reliability, i.e., the probability that the shortest transmission time from the source node s to the sink node t is less than or equal to d and the total transmission cost is no more than c, can then be computed in terms of these (d,c)-MCs.

Reference

[1] K. K. Aggarwal, J. S. Gupta and K. G. Misra, ”A Simple Method for Reliability Evaluation of a Communication System,” IEEE Transactions on Communication, Vol. 23, pp. 563-565, 1975.

[2] T. Aven, “Reliability Evaluation of Multistate Systems with Multistate Components,” IEEE Transactions on Reliability, Vol. 34, pp. 473-479, 1985.

[3] T. Aven, “Same Considerations on Reliability Theory and its Applications,” Reliability Engineering and System Safety, Vol. 21, pp. 215-223, 1988.

[4] G. Bai, Z. Tian, and M. J. Zuo, “An improved algorithm for finding all minimal paths in a network,” Reliability Engineering and System Safety, Vol. 150, pp. 1-10, 2016.

[5] M. O. Ball, “Computational Complexity of Network Reliability Analysis: An overview,” IEEE Transactions on Reliability, Vol. 35, pp. 230-238, 1986.

[6] R. E. Barlow and A. S. Wu, “Coherent Systems with Multistate Components,” Mathematics of Operations Research, Vol. 3, pp. 275-281, 1978.

[7] S. G. Chen and Y. K. Lin, “Search for all minimal paths in a general large flow network,” IEEE Transactions on Reliability, Vol. 61, pp. 949-956, 2012.

[8] E. El-Neweihi, F. Proschan, and J. Sethuraman, “Multistate Coherent Systems,” Journal of Applied Probability, Vol. 15, pp. 675-688, 1978.

[9] W. S. Griffith, “Multistate Reliability Models,” Journal of Applied Probability, Vol. 17, pp. 735-744, 1980.

[10] J. C. Hudson and K. C. Kapur, “Reliability Bounds for Multistate Systems with Multistate Components,” Operations Research Vol. 33, pp. 153-160, 1985.

[11] C. C. Jane, J. S. Lin, and J. Yuan, “Reliability Evaluation of a Limited-Flow Network in Terms of Minimal

ISSN:2319-7900

Cutsets," IEEE Transactions on Reliability, Vol. 42, pp. 354-361, 1993.
[12] J. S. Lin," Reliability Evaluation of Capacitated-Flow Networks with Budget Constraints," IIE Transactions Vol. 30, pp. 1175-1180, 1998.
[13] J. S. Lin, C. C. Jane, and J. Yuan," On Reliability Evaluation of a Capacitated-Flow Network in Terms of Minimal Pathsets," Networks, Vol. 25, pp. 131-138, 1995.
[14] W. C. Yeh, "Search for Minimal Paths in Modified Networks," Reliability Engineering and System Safety, Vol. 75, pp. 389-395, 2007.
[15] W. C. Yeh, "A Simple Universal Generating Function Method to Search for All Minimal Paths in Networks," IEEE Transactions on System, Man and Cybernetics, Vol. 39, pp. 1247-1254, 2009.

Biographies

Jsen-Shung Lin received the B.S. degree in Department of Mathematics, the M.S. and Ph.D. degrees in Department of Industrial Engineering from National Tsing Hua University, Hsinchu city, Taiwan, respectively. He is currently an Associative Professor and Chairman of the Department of Information Management at Central Police University in Taiwan. His research interests include network reliability theory, graph theory, linear programming, and decision making.