International Journal of Advanced Computer Technology (IJACT)
ISSN:2319-7900

K-REGULAR GRAPH WITH DIAMETER 2

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Abstract

A regular graph with vertices of degree k is called a k - regular graph or regular graph of deree k. G is said to be strongly regular if there are also integers τ and θ such that: every two adjacent vertices have τ common neighbours and every two non-adjacent vertices have θ common neighbours. A graph of this kind is sometimes said to be an $SRG(n,k,\tau,\theta)$. The length $\max_{(u,v)} d(u,v)$ of the "longest shortest path" (i.e., the longest graph geodesic) between any two vertices (u,v) of a graph, where $d(u,v)$ is a graph distance. n_k denotes the maximum number of vertices in a k - regular graph with diameter 2. We will prove that $5(k-1) \le n_k \le k^2 + 1$ and determine n_k for some special values of k.

Keywords: k - regular graph, strongly regular graph, diameter.

Introduction

A graph G is a pair (V,E), where E is a set of pairs of V (V is called vertex - set). The neighborhood of a vertex v denoted by $N(v)$, i.e., $N(v) = \{u \in V : (u,v) \in E\}$. Note that $v \notin N(v)$. The size of $N(v)$ is called the degree of v, $\deg(v)$. The graph G has diameter 2 if it is not the complete graph and for each two vertices $u,v \in V$ either (u,v) is an edge of G, or $N(u) \cap N(v) \ne \varnothing$ (or both). G is said to be strongly regular if there are also integers τ and θ such that: every two adjacent vertices have τ common neighbours and every two non-adjacent vertices have θ common neighbours. A graph of this kind is sometimes said to be an $SRG(n,k,\tau,\theta)$ [1]. Example, C_5 is in $SRG(5,2,0,1)$. Petersen Graph is in $SRG(10,3,0,1)$ and it is composed of two cycles (a_1,a_2,a_3,a_4,a_5) and (b_1,b_2,b_3,b_4,b_5), it is added by edges $(a_1,b_1),(a_2,b_3),(a_3,b_5),(a_4,b_2),(a_5,b_4)$.

n_k denotes the maximum number of vertices in a k - regular graph with diameter 2. In this paper, we will prove that $5(k-1) \le n_k \le k^2 + 1$ and determine n_k for some special values of k.

2 Main Results

In [4], Zoltan Furedi evaluated for the smallest number edges of a k - regular graph with diameter 2.

In [3], the authors show that with the exception of C_4, there are no graphs of diameter 2, of maximum degree d, and with d^2 vertices.

Theorem 1 (Paul Erdös,Siemion Fajtlowicz,Alan J. Hoffman) *If G is a graph of diameter 2 with $n = d^2$ ($d \ge 2$) vertices and maximum degree d, then G is isomorphic to a four-element cycle.*

In our papers, we will estimate the maximum number of vertices in a k - regular graph with diameter 2.

We prove that $n_{k+1} \ge 5k$ by constructing a graph have $5k$ vertices, $k+1$ - regular, $P(k+1) = (V,E)$ with diameter 2 and

$$V = \bigcup_{i=1}^{k} X_i, \quad X_i = \{a_{5i-4}, a_{5i-3}, a_{5i-2}, a_{5i-1}, a_{5i}\}.$$

When $k = 2$, $P(3)$ is a Petersen graph.

E have edges:

(i) $(a_{5i-4},a_{5i-3}),(a_{5i-3},a_{5i-2}),(a_{5i-2},a_{5i-1}),(a_{5i-1},a_{5i}),(a_{5i},a_{5i-4})$ $\forall i = \overline{1,k}$.

(ii) (a_{5i-4},a_{5j-4}), (a_{5i-3},a_{5j-1}), (a_{5i-2},a_{5j-3}), (a_{5i-1},a_{5j}), (a_{5i},a_{5j-2}) $\forall i,j = \overline{1,k}$, $i < j$.

Obviously, $P(k+1)$ is the graph with $5k$ vertices and each vertex in the set X_i connects to only vertex in the set X_j ($i \ne j$) and 2 edges with the same vertex set, so the degree of each vertex of $P(k+1)$ is $k+1$. Here, we will prove that $P(k+1)$ is the graph with diameter 2.

Lemma 1 *$P(k+1)$ is a $k+1$ - regular graph on $5k$ vertices with diameter 2.*

Proof. Let u,v be two non-adjacent vertices in the set V. We prove that $N(u) \cap N(v) \ne \varnothing$.

We consider two cases:

+ **Case 1**: $\exists i$ such that $u,v \in X_i$.

ISSN:2319-7900

If $u = a_{5i-4}$, then v is in $\{a_{5i-2}, a_{5i-1}\}$ (Because u, v are non-adjacent and $N(a_{5i-4}) \cap N(a_{5i-2}) = \{a_{5i-3}\}$, $N(a_{5i-4}) \cap N(a_{5i-1}) = \{a_{5i}\}$).
Thus, $u = a_{5i-4}$ satisfies the condition.
Similar to the cases: $u = a_{5i-3}, a_{5i-2}, a_{5i-1}, a_{5i}$.
+ **Case 2**: $\exists i < j$ such that $u \in X_i$ and $v \in X_j$.
If $u = a_{5i-4}$, then $v = a_{5j-3}, a_{5j-2}, a_{5j-1}, a_{5j}$,
When $v = a_{5j-3}$, we obtain $N(u) \cap N(v) = \{a_{5j-4}\}$,
When $v = a_{5j-2}$, we obtain $N(u) \cap N(v) = \{a_{5i}\}$,
When $v = a_{5j-1}$, we obtain $N(u) \cap N(v) = \{a_{5i-3}\}$,
When $v = a_{5j}$, we obtain $N(u) \cap N(v) = \{a_{5j-4}\}$,
Thus, $u = a_{5i-4}$ satisfies the condition.
Similar to the cases: $u = a_{5i-3}, a_{5i-2}, a_{5i-1}, a_{5i}$.
Hence, we have $N(u) \cap N(v) \neq \varnothing$ in all cases.

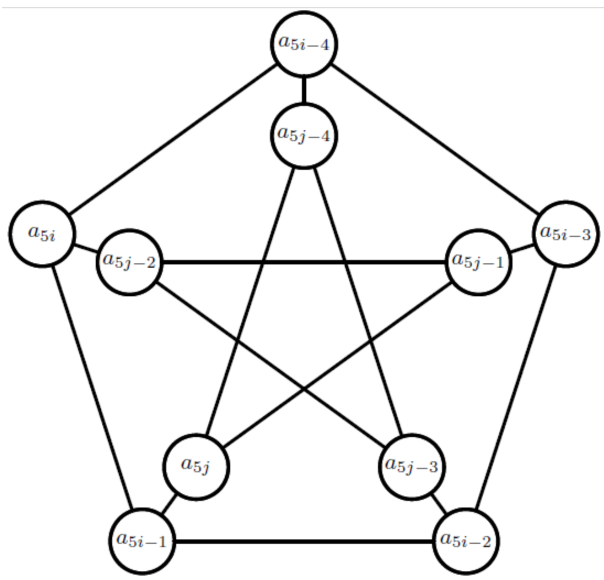


Figure 1. Edges of X_i and X_j

First, we construct the structure of V in graph G with diameter 2 in the following lemma:

Lemma 2 *Suppose $G = (V, E)$ is a graph with diameter 2. Let $x \in V$ be an arbitrary vertex and $N(x) = \{y_1, y_2, \cdots, y_k\}$. We have*
$V = \{x\} \cup N(x) \cup N(y_1) \cup N(y_2) \cup \cdots \cup N(y_k)$

Proof. Obviously,
$\left(\{x\} \cup N(x) \cup N(y_1) \cup N(y_2) \cup \cdots \cup N(y_k)\right) \subset V$.
Suppose
$\exists y \in V - \left(\{x\} \cup N(x) \cup N(y_1) \cup N(y_2) \cup \cdots \cup N(y_k)\right)$.

Because $y \notin N(x)$, y is not adjacent to x, G have diameter 2 so that $w \in N(x) \cap N(y)$. Because w is adjacent to x, $w \in N(x) = \{y_1, y_2, \cdots, y_k\}$ and $y \in N(w)$, this is a contradiction ($y \notin \bigcup_{i=1}^{k} N(y_i)$).

Thus,
$V = \{x\} \cup N(x) \cup N(y_1) \cup N(y_2) \cup \cdots \cup N(y_k)$.

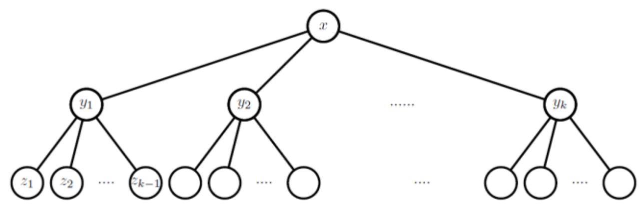


Figure 2. Vertices of k-regular graph with diameter 2.

We have a theorem to estimate n_k as follow:

Theorem 2 *When $k \in \mathbb{N}^*$, we have*
$$5(k-1) \le n_k \le k^2 + 1.$$
Proof. By Theorem 1, we have $P(k)$ is a k - regular graph with diameter 2, i.e. $n_k \ge 5(k-1)$.
On the other hand, by Theorem 2 we have
$$|V| = \left|\{x\} \cup N(x) \cup (N(y_1) - \{x\}) \cup (N(y_2) - \{x\}) \cup \cdots \cup (N(y_k) - \{x\})\right|$$
$$\le \left|\{x\}\right| + \left|N(x)\right| + \sum_{i=1}^{k} |N(y_i) - \{x\}| = 1 + k + k(k-1) = k^2 + 1$$
.

In [2], Hoffman and Singleton proved this result as follow:

Theorem 3 *$G(r^2+1; r; 0; 1)$ exists when $r = 2; 3; 7$ and possible $r = 57$. $G(50; 7; 0; 1)$ graph, Hoffman Singleton graph, if there exists, it is unique.*

By estimating n_k in Theorem 2, we have $n_k = k^2 + 1$ for $n = 2, 3, 7$, and if there exists Hoffman Singleton graph, then $n_{57} = 57^2 + 1$. In next Theorem, we determine n_4.

Theorem 4 $n_2 = 5, n_3 = 10$ *and* $n_4 = 15$.

ISSN:2319-7900

Proof. By Theorem 2, we obtain $5=5(2-1)\le n_2\le 2^2+1=5$. Hence, $n_2=5$. And $P(1)$ is a 2 - regular graph with diameter 2.

By Theorem 2, we have $10=5(3-1)\le n_2\le 3^2+1=10$, thus $n_3=10$. And $P(2)$ is a 3 - regular graph with diameter 2.

By Theorem 2, we obtain $15=5(4-1)\le n_2\le 4^2+1=17$.

We will prove that $n_4\ne 17$ and $n_4\ne 16$.

1. Prove that $n_4\ne 17$:

Indeed, suppose $n_4=17$, thus there exists $G=(V,E)$ so that $|V|=17$ and G is a 4 - regular graph with diameter 2.

Let vertex x be fix in V. Because $\deg(x)=4$, $N(x)=\{y_1,y_2,y_3,y_4\}$.

By theorem 2, we have $V=\{x\}\cup N(x)\cup\left(\bigcup_{i=1}^{k}X_i\right)$, where $X_i=N(y_i)-(\{x\}\cup N(x))$.

If $(y_i,y_j)\in E$ (where $i\ne j$).

Without loss of generality, suppose y_1 is adjacent to y_2.

Hence, $N(y_1)=\{x,y_2,a,b\}$, $N(y_2)=\{x,y_1,c,d\}$.

Thus, $X_1\subset\{a,b\},X_2\subset\{c,d\}$. Hence, $|X_1|\le 2$, $|X_2|\le 2$. Obviously, $|X_3|\le 3$, $|X_4|\le 3$.

Finally, $|V|\le 1+4+2+2+3+3=15$. This is a contradiction ($|V|=17$).

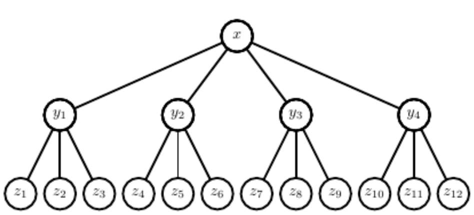
Thus, y_i is not adjacent to y_j (where $i\ne j$).

So that $X_i=N(y_i)-\{x\}$ and $|X_i|=3$.

$$17=|V|\le 1+4+\sum_{i=1}^{4}|X_i|=5+3.4=17.$$

So the equal sign must occur in the above inequality. Hence, $X_i\cap X_j=\varnothing$.

On other hand, $|X_i|=3$. Hence $X_i=\{z_{3i-2},z_{3i-1},z_{3i}\}$ where $i=\overline{1,4}$.



+) We observe that if $u\in X_i$, then u is not adjacent to y_j ($j\ne i$) and x.

We prove that $u\in X_i$, then there exists unique edge of vertex u and one vertex in X_j ($i\ne j$). Without loss of generality, we consider the case $i=1, j=2$.

Indeed, if u is adjacent to two vertices $v,w\in X_2$, then $N(u)=\{y_1,v,w,a\}$ where a is a vertex.

Because u is not adjacent to two vertices y_3,y_4, we have $a\in N(y_3)\cap N(y_4)=\{x\}$ (vertices $y_1,v,w\notin N(y_3),N(y_4)$). So that $a=x$ and u is adjacent to x.

Hence, $\{u,y_1,y_2,y_3,y_4\}\subset N(x)\Rightarrow |N(x)|\ge 5$

(A contradiction with $|N(x)|=4$).

If u is not adjacent to y_2, $N(u)\cap N(y_2)\ne\varnothing$ and so that u must be adjacent to a vertex in $N(y_2)=\{x,z_4,z_5,z_6\}$. But u is not adjacent to x, u is adjacent to a vertex in $X_2=\{z_4,z_5,z_6\}$.

Finally, if $u\in X_i$, then u is adjacent to only one vertex in X_j ($j\ne i$). (1)

+) We will prove that if $u,v\in X_i$ and $u\ne v$, then u,v is non-adjacent.

Without loss of gerenality, we only consider the case $i=1$.

By (1) and (2) u must only be adjacent to the vertices $a\in X_2$, $b\in X_3$, $c\in X_4$

Hence $\{v,y_1,a,b,c\}\subset N(u)$, $|N(u)|\ge 5$. A contradiction with $|N(u)|=4$.

Finally, two vertices in X_i are non - adjacent. (2)

+) We consider vertex z_1 :

By (1) and (2), we have $N(z_1)=\{y_1,a,b,c\}$ where $a\in X_2,b\in X_3,c\in X_4$. Without loss of generality, we consider the case $a=z_4,b=z_7,c=z_{10}$ (if not we can renumber the elements in X_i, $i=2,3,4$).

Hence $N(z_1)=\{y_1,z_4,z_7,z_{10}\}$.

+) We consider vertex z_2 :

$N(z_2)=\{y_1,a,b,c\}$ where $a\in X_2,b\in X_3,c\in X_4$. We obtain $a\ne z_4$ because if $a=z_4$, then z_4 is adjacent to two vertices z_1,z_2 of X_1 (a contradiction). Therefore z_2 is adjacent to z_5,z_6. Without loss of generality, we only consider the case $a=z_5$. Similarly, $b=z_8$, $c=z_{11}$.

Finally, $N(z_2)=\{y_1,z_5,z_8,z_{11}\}$.

+) The same with vertex z_3, we have $N(z_3)=\{y_1,z_6,z_9,z_{12}\}$.

+) We consider vertex z_4: $N(z_4)=\{y_2,z_1,a,b\}$.

ISSN:2319-7900

z_4 is adjacent to z_1, i.e. z_4 is not adjacent to z_2, z_3, y_3, y_4, x.

$N(y_3) = \{x, z_7, z_8, z_9\}$, $N(y_4) = \{x, z_{10}, z_{11}, z_{12}\}$.

$$\text{Hence } \left[\begin{array}{l} z_4 \text{ is adjacent to } z_8, z_{12} \\ z_4 \text{ is adjacent to} z_9, z_{11} \end{array}\right.$$

Without loss of generality, we consider the case z_4 is adjacent to z_8, z_{12}.

So that $N(z_4) = \{y_2, z_1, z_8, z_{12}\}$.

+ We consider vertex z_5: $N(z_5) = \{y_2, z_2, a, b\}$.

z_5 is not adjacent to z_1, z_3, y_3, y_4, x so that

$$\left[\begin{array}{l} z_5 \text{ is adjacent to } z_7, z_{12} \\ (\text{A contradiction, because } z_4 \text{ is adjacent to } z_{12}) \\ z_5 \text{ is adjacent } z_{10}, z_9 \end{array}\right.$$

Therefore $N(z_5) = \{y_2, z_2, z_9, z_{10}\}$.

+ We consider vertex z_6: $N(z_6) = \{y_2, z_3, a, b\}$.

z_6 is not adjacent to z_8, z_9. Because z_8 is adjacent to z_4, z_9 is adjacent to z_5, z_6 is adjacent to z_7. Similarly, z_6 is adjacent z_{11}.

Consequently, $N(z_6) = \{y_2, z_2, z_7, z_{11}\}$.

+ We consider vertex z_7: $N(z_7) = \{y_3, z_1, z_6, a\}$.

Where $a \in X_4$. Because z_7 is adjacent to z_2, z_3, z_4, z_5 and $N(z_2) = \{y_1, z_5, z_8, z_{11}\}$, we have $a = z_{11}$.

So that $N(z_7) = \{y_3, z_1, z_6, z_{11}\}$.

On other hand, we have $N(z_3) = \{y_1, z_6, z_9, z_{12}\}$.

Hence $N(z_3) \cap N(z_7) = \varnothing$, but z_3, z_7 are non-adjacent. A contradiction with G have diameter 2.

There doesn't exist 4 - regular, 2 - diameter graph with 17 edges.

2. Prove that $n_4 \neq 16$:

Suppose $n_4 = 16$, hence there exists a graph $G = (V, E)$ with 2 - diameter, 4 - regular and $|V| = 16$.

Because G is a 4 - regular graph, by Theorem 2:

$$V = (\bigcup_{i=1}^{4} N(y_i)) \cup \{x\} \cup N(x),$$

where $N(x) = \{y_1, y_2, y_3, y_4\}$ and $N(y_i) = \{x, a, b, c\}$.

If $y_j \in \{a, b, c\}$ (where $i \neq j$), prove that similarly, we have $|V| \leq 15$ (A contradiction with $|V| = 16$).

Consequently, y_i is not adjacent to y_j (where $i \neq j$).

Let $X_i = N(y_i) - (\{x\} \cup N(x))$. As above, we have $|X_i| = 3$ for all $i = \overline{1,4}$.

Let $\alpha = \max\{|X_i \cap X_j| \,|\, i \neq j, i, j = \overline{1,4}\}$.

If $\alpha = 0$, then $|X_i \cap X_j| = 0$ for all $i \neq j$. Hence, $X_i \cap X_j = \varnothing$ for all $i \neq j$.

On other hand, $V = (\bigcup_{i=1}^{4} X_i) \cup \{x\} \cup N(x)$. Therefore, we have:

$$|V| = |\bigcup_{i=1}^{4} X_i| + 1 + 4 = \sum_{i=1}^{4} |X_i| + 5 = 4.3 + 5 = 17.$$

(A contradiction with $|V| = 16$).

So that $\alpha \geq 1$. Suppose $|X_3 \cap X_4| = \alpha$.

$$16 = |V| = |\bigcup_{i=1}^{4} X_i| + 5 \leq |X_1 \cup X_2| + |X_3 \cup X_4| + 5,$$

$$\leq 11 + |X_3 \cup X_4| = 11 + |X_3| + |X_4| - |X_3 \cap X_4| = 17 - \alpha$$

hence $\alpha \leq 1$.

Finally, $\alpha = 1$ and the equal sign must occur in the inequation:

So that $(X_1 \cup X_2) \cap (X_3 \cup X_4) = \varnothing$ and $X_1 \cap X_2 = \varnothing$, $|X_3 \cap X_4| = 1$. Thus, $X_1 = \{z_1, z_2, z_3\}$, $X_2 = \{z_4, z_5, z_6\}$, $X_3 = \{z_7, z_8, z_9\}$, $X_4 = \{z_9, z_{10}, z_{11}\}$.

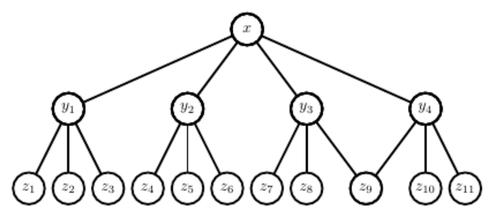


Figure 3. When n_4=16.

+ We consider vertex z_9: we have $N(z_9) = \{y_3, y_4, a, b\}$.

Because z_9 is not adjacent to y_1, z_9 is adjacent to z_1, z_2, z_3. Without loss of generality, we consider the case which z_9 is adjacent to z_{14}.

Similarly, z_9 is adjacent to z_4.

Consequently, $N(z_9) = \{y_3, y_3, z_1, z_4\}$.

+ We consider vertex z_2: $N(z_2) = \{y_1, a, b, c\}$.

Because z_2 is not adjacent to z_9 (where $N(z_9) = \{y_3, y_4, z_1, z_4\}$), we obtain $\{a, b, c\} \cap \{z_1, z_4\} \neq \varnothing$.

ISSN:2319-7900

Suppose z_2 is adjacent to z_1, we have z_2 is not adjacent to y_2, y_3, y_4.

$N(y_2) = \{x, z_4, z_5, z_6\}$, $N(y_3) = \{x, z_7, z_8, z_9\}$,

$N(y_4) = \{x, z_9, z_{10}, z_{11}\}$.

Hence z_2 will be adjacent to one of three vertices z_4, z_5, z_6, this vertex is called by a, z_2 is adjacent to one of three vertices z_7, z_8, z_9, this vertex is called by b ($b \neq z_9$ because $z_2 in N(z_9)$). Similarly, z_2 is adjacent to one of two vertices z_{10}, z_{11} and this vertex is called by c. Therefore, we obtain $\{y_1, z_1, a, b, c\} \subset N(z_2) \Rightarrow |N(z_2)| \geq 5$.

(A contradiction with $|N(z_2)| = 4$)

We have z_2 is not adjacent to z_1, hence z_2 is adjacent to z_4.

Because z_2 is not adjacent to y_3, z_2 is adjacent to one of three vertices z_7, z_8, z_9. Therefore, it is adjacent to one of two vertices z_7, z_8. Without loss of generality, we consider the case which z_2 is adjacent to z_7. Similarly, z_2 is adjacent to $z_1 0$. Consequently, $N(z_2) = \{y_1, z_4, z_7, z_{10}\}$.

+ We consider vertex z_3:

Similarly as z_2, z_3 is adjacent to z_4 and z_3 is adjacent to one of two vertices z_7, z_8.

Case z_3 is adjacent to z_7:

Because z_7 is not adjacent to y_4, z_7 is adjacent to one of two vertices z_{10}, z_{11}, this vertex is called by a.

z_7 is adjacent to one of three vertices z_4, z_5, z_6, called by b.

$\Rightarrow \{y_2, z_3, a, b, z_2\} \subset N(z_7) \Rightarrow |N(z_7)| \geq 5$.

(A contradiction with $|N(z_7))| = 4$)

Consequently, z_3 must be adjacent to z_8. Similarly, z_3 is adjacent to z_{11}.

Finally, $N(z_3) = \{y_1, z_4, z_8, z_{11}\}$.

+ We consider vertex z_5:

Similarly as z_2, we obtain z_5 is adjacent to z_1 and z_6 is adjacent to z_1.

Because z_5 is not adjacent to y_3, z_5 is adjacent to one of two vertices z_7, z_8.

Case z_5 is adjacent to z_7:

z_5 is adjacent to z_{11} (Because z_5 is not adjacent to z_3, y_4)

Case z_5 is adjacent to z_8:

z_5 is adjacent to z_{10} (Because z_5 is not adjacent to z_2, y_4)

Without loss of generality, we consider the case which z_5 is adjacent to z_7, z_{11}

Therefore z_6 is adjacent to z_8, z_{10}.

Consequently, $N(z_5) = \{y_2, z_1, z_7, z_{11}\}$ and $N(z_6) = \{y_2, z_1, z_8, z_{10}\}$.

+) We consider vertex z_7:

We have $N(z_7) = \{y_3, z_2, z_5, a\}$.

Because z_7 is not adjacent to y_4, $N(y_4) = \{x, z_9, z_{10}, z_{11}\}$, $\{a\} \cap \{z_{10}, z_{11}\} \neq \emptyset$.

Because z_7 is not adjacent to z_3 ($N(z_3) = \{y_1, z_4, z_8, z_{11}\}$), $\{a\} \cap \{z_8, z_{11}\} \neq \emptyset$.

So that $a = z_{11}$.

Hence $N(z_7) = \{y_3, z_2, z_5, z_{11}\}$.

Because z_7 is not adjacent to z_6 ($N(z_6) = \{y_2, z_1, z_8, z_{10}\}$), $N(z_6) \cap N(z_7) = \emptyset$. A contradiction with diameter of G =2.

Consequently, $n_4 \neq 16$.

Finally, $n_4 \leq 15$ and $n_4 = 15$. We have $P(3)$ with $|V| = 15$, and $P(3)$ is 4-regular graph with diameter 2.

3. Conclusion

In this paper, we define n_k which is maximum number of vertices in a k - regular graph with diameter 2, and we estimate n_k for all $k \in$ * and determine n_k in case $k = 2, 3, 4$.

Acknowledgments

This research was supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED), No. 102.01-2012.29.

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