K-REGULAR GRAPH WITH DIAMETER 2

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Abstract

A regular graph with vertices of degree \( k \) is called a \( k \)-regular graph or regular graph of degree \( k \). \( G \) is said to be strongly regular if there are also integers \( \tau \) and \( \theta \) such that: every two adjacent vertices have \( \tau \) common neighbours and every two non-adjacent vertices have \( \theta \) common neighbours. A graph of this kind is sometimes said to be a \( SRG(n,k,\tau,\theta) \). The length \( \max_d(u, v) \) of the "longest shortest path" (i.e., the longest geodesic) between any two vertices \((u, v)\) of a graph, where \( d(u, v) \) is a graph distance. \( n_k \) denotes the maximum number of vertices in a \( k \)-regular graph with diameter 2. We will prove that \( 5(k-1) \leq n_k \leq k^2 + 1 \) and determine \( n_k \) for some special values of \( k \).

Keywords: \( k \)-regular graph, strongly regular graph, diameter.

Introduction

A graph \( G \) is a pair \((V,E)\), where \( E \) is a set of pairs of \( V \) (\( V \) is called vertex set). The neighborhood of a vertex \( v \) denoted by \( N(v) \), i.e., \( N(v) = \{ u \in V : (u,v) \in E \} \). Note that \( v \notin N(v) \). The size of \( N(v) \) is called the degree of \( v \), \( \deg(v) \). The graph \( G \) has diameter 2 if it is not the complete graph and for each two vertices \( u, v \in V \) either \((u,v)\) is an edge of \( G \), or \( N(u) \cap N(v) = \emptyset \) (or both). \( G \) is said to be strongly regular if there are also integers \( \tau \) and \( \theta \) such that: every two adjacent vertices have \( \tau \) common neighbours and every two non-adjacent vertices have \( \theta \) common neighbours. A graph of this kind is sometimes said to be an \( SRG(n,k,\tau,\theta) \) [1]. Example, \( C_3 \) is in \( SRG(5,2,0,1) \). Petersen Graph is in \( SRG(10,3,0,1) \) and it is composed of two cycles \((a_1,a_2,a_3,a_4,a_5)\) and \((b_1,b_2,b_3,b_4,b_5)\), it is added by edges \((a_1,b_1),(a_2,b_2),(a_3,b_3),(a_4,b_4),(a_5,b_5)\).

\( n_k \) denotes the maximum number of vertices in a \( k \)-regular graph with diameter 2. In this paper, we will prove that \( 5(k-1) \leq n_k \leq k^2 + 1 \) and determine \( n_k \) for some special values of \( k \).

2 Main Results

In [4], Zoltan Furedi evaluated for the smallest number edges of a \( k \)-regular graph with diameter 2.

In [3], the authors show that with the exception of \( C_4 \), there are no graphs of diameter 2, of maximum degree \( d \), and with \( d^2 \) vertices.

Theorem 1 (Paul Erdös, Siemion Fajtlowicz, Alan J. Hoffman) If \( G \) is a graph of diameter 2 with \( n = d^2 \) (\( d \geq 2 \)) vertices and maximum degree \( d \), then \( G \) is isomorphic to a four-element cycle.

In our papers, we will estimate the maximum number of vertices in a \( k \)-regular graph with diameter 2.

We prove that \( n_{k+1} \geq 5k \) by constructing a graph have \( 5k \) vertices, \( k+1 \)-regular, \( P(k+1) = (V,E) \) with diameter 2 and
\[ V = \bigcup_{i=1}^{k} X_i, \quad X_i = \{a_{5i-4}, a_{5i-3}, a_{5i-2}, a_{5i-1}, a_{5i}\}. \]

When \( k = 2 \), \( P(3) \) is a Petersen graph. \( E \) have edges:
(i) \((a_{5i-4}, a_{5i-3}), (a_{5i-3}, a_{5i-2}), (a_{5i-2}, a_{5i-1}), (a_{5i-1}, a_{5i}), (a_{5i}, a_{5i-4})\) for \( i = 1, k \).
(ii) \((a_{5i-4}, a_{5j-4}), (a_{5i-3}, a_{5j-3}), (a_{5i-2}, a_{5j-2}), (a_{5i-1}, a_{5j}), (a_{5i}, a_{5j-2})\) for \( i,j = 1, k \), \( i < j \).

Obviously, \( P(k+1) \) is the graph with \( 5k \) vertices and each vertex in the set \( X_i \) connects to only vertex in the set \( X_j \) (\( i \neq j \)) and 2 edges with the same vertex set, so the degree of each vertex of \( P(k+1) \) is \( k+1 \). Here, we will prove that \( P(k+1) \) is the graph with diameter 2.

Lemma 1 \( P(k+1) \) is a \( k+1 \)-regular graph on \( 5k \) vertices with diameter 2.

Proof. Let \( u, v \) be two non-adjacent vertices in the set \( V \).

We prove that \( N(u) \cap N(v) = \emptyset \).

We consider two cases:

\[ \text{Case 1: } \exists i \text{ such that } u,v \in X_i. \]
If \( u = a_{i_4} \), then \( v \) is in \( \{a_{i_2}, a_{i_4}\} \) (Because \( u, v \) are non-adjacent and \( N(a_{i_4}) \cap N(a_{i_2}) = \{a_{i_2}\} \)).

Thus, \( u = a_{i_4} \) satisfies the condition.

Similar to the cases: \( u = a_{i_4}, a_{i_4}, a_{i_4}, a_{i_4} \).

\[ \text{Case 2:} \exists i < j \text{ such that } u \in X_i \text{ and } v \in X_j. \]

If \( u = a_{i_4} \), then \( v = a_{i_4}, a_{i_4}, a_{i_4}, a_{i_4} \).

When \( v = a_{i_4} \), we obtain \( N(u) \cap N(v) = \{a_{i_4}\} \).

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Thus, \( u = a_{i_4} \) satisfies the condition.

Similar to the cases: \( u = a_{i_4}, a_{i_4}, a_{i_4}, a_{i_4} \).

Hence, we have \( N(u) \cap N(v) \neq \emptyset \) in all cases.

**Proof.** Obviously, \( \{x \cup N(x) \cup N(y_1) \cup N(y_2) \cup \ldots \cup N(y_k)\} \subset V \).

Suppose \( \exists y \in V - \{x \cup N(x) \cup N(y_1) \cup N(y_2) \cup \ldots \cup N(y_k)\} \).

Because \( y \not\in N(x) \), \( y \) is not adjacent to \( x \), \( G \) have diameter 2 so that \( w \in N(x) \cap N(y) \). Because \( w \) is adjacent to \( x \), \( w \in N(x) = \{y_1, y_2, \ldots, y_k\} \) and \( y \in N(w) \), this is a contradiction ( \( y \notin \bigcup_{i=1}^{k} N(y_i) \)).

Thus, \( V = \{x \cup N(x) \cup N(y_1) \cup N(y_2) \cup \ldots \cup N(y_k)\} \).

**Figure 2. Vertices of \( k \)-regular graph with diameter 2.**

We have a theorem to estimate \( n_k \) as follow:

**Theorem 2** When \( k \in \mathbb{N} \), we have \( 5(k-1) \leq n_k \leq k^2 + 1 \).

**Proof.** By Theorem 1, we have \( P(k) \) is a \( k \)-regular graph with diameter 2, i.e. \( n_k \geq 5(k-1) \).

On the other hand, by Theorem 2 we have

\[
\begin{align*}
|V| &= |x| + |N(x) \cup (N(y_1) - \{x\}) \cup (N(y_2) - \{x\}) \cup \ldots \cup (N(y_k) - \{x\})| \\
&\leq |x| + |N(x)| + \sum_{i=1}^{k} |N(y_i) - \{x\}| = 1 + k + k(k-1) = k^2 + 1
\end{align*}
\]

In [2], Hoffman and Singleton proved this result as follow:

**Theorem 3** \( G(r^2 + 1; r; 0; 1) \) exists when \( r = 2; 3; 7 \) and possible \( r = 57 \). \( G(50; 7; 0; 1) \) graph, Hoffman Singleton graph, if there exists, it is unique.

By estimating \( n_k \) in Theorem 2, we have \( n_k = k^2 + 1 \) for \( n = 2, 3, 7 \), and if there exists Hoffman Singleton graph, then \( n_{57} = 57^2 + 1 \). In next Theorem, we determine \( n_k \).

**Theorem 4** \( n_2 = 5, n_3 = 10 \) and \( n_4 = 15 \).
Proof. By Theorem 2, we obtain 
\[ 5 = 5(2 - 1) \leq n_2 \leq 2^2 + 1 = 5. \] Hence, \( n_2 = 5 \). And \( P(1) \) is a 2 - regular graph with diameter 2.

By Theorem 2, we have \( 10 = 5(3 - 1) \leq n_2 \leq 3^2 + 1 = 10 \), thus \( n_3 = 10 \). And \( P(2) \) is a 3 - regular graph with diameter 2.

By Theorem 2, we obtain \( 15 = 5(4 - 1) \leq n_2 \leq 4^2 + 1 = 17 \). We will prove that \( n_4 \neq 17 \) and \( n_4 \neq 16 \).

1. Prove that \( n_4 \neq 17 \):

Indeed, suppose \( n_4 = 17 \), thus there exists \( G = (V, E) \) so that \( |V| = 17 \) and \( G \) is a 4 - regular graph with diameter 2.

Let vertex \( x \) be fix in \( V \). Because \( \deg(x) = 4 \), \( N(x) = \{y_1, y_2, y_3, y_4\} \).

By theorem 2, we have \( V = \{x\} \cup N(x) \cup \bigcup_{i=1}^{k} X_i \), where 
\[
X_i = N(y_i) - \{x\} \cup N(x). 
\]

If \((y_i, y_j) \in E \) (where \( i \neq j \)).

Without loss of generality, suppose \( y_1 \) is adjacent to \( y_2 \). Hence, \( N(y_1) = \{x, y_1, a, b\} \) and \( N(y_2) = \{x, y_2, c, d\} \).

Thus, \( X_1 \subset \{a, b\}, X_2 \subset \{c, d\} \). Hence, \( |X_1| \leq 2, |X_2| \leq 2 \). Obviously, \( |X_1| \leq 3 \), \( |X_2| \leq 3 \).

Finally, \( |V| \leq 1 + 4 + 2 + 2 + 3 + 3 = 15 \). This is a contradiction \(|V| = 17\).

Thus, \( y_i \) is not adjacent to \( y_j \) (where \( i \neq j \)).

So that \( X_i = N(y_i) - \{x\} \) and \( |X_i| = 3 \).

\[ 17 = |V| \leq 1 + 4 + \sum_{i=1}^{4} |X_i| \leq 5 + 3.4 = 17. \]

So the equal sign must occur in the above inequality. Hence, \( X_i \cap X_j = \emptyset \).

On other hand, \( |X_i| = 3 \). Hence \( X_i = \{z_{i-2}, z_{i-1}, z_i\} \) where \( i = 1, 4 \).

We prove that \( u \in X_j \), then there exists unique edge of vertex \( u \) and one vertex in \( X_j \) (\( i \neq j \)). Without loss of generality, we consider the case \( i = 1, j = 2 \).

Indeed, if \( u \) is adjacent to two vertices \( v, w \in X_2 \), then \( N(u) = \{y_1, v, w\} \) where \( a \) is a vertex.

Because \( u \) is not adjacent to two vertices \( y_3, y_4 \), we have \( a \in N(y_3) \cap N(y_4) = \{x\} \) (vertices \( y_1, v, w \notin N(y_3), N(y_4) \)). So that \( a = x \) and \( u \) is adjacent to \( x \).

Hence, \( \{u, y_1, y_2, y_3, y_4\} \subset N(x) \Rightarrow |N(x)| \geq 5 \) (A contradiction with \( |N(x)| = 4 \)).

If \( u \) is not adjacent to \( y_2 \), \( N(u) \cap N(y_2) \neq \emptyset \) and so that \( u \) must be adjacent to a vertex in \( N(y_2) = \{x, z_4, z_5, z_6\} \). But \( u \) is not adjacent to \( x \), \( u \) is adjacent to a vertex in \( X_2 = \{z_4, z_5, z_6\} \).

Finally, if \( u \in X_1 \), then \( u \) is adjacent to only one vertex in \( X_j \) (\( j \neq i \)). (1)

+) We will prove that if \( u, v \in X_i \) and \( u \neq v \), then \( u, v \) is non-adjacent.

Without loss of generality, we only consider the case \( i = 1 \).

By (1) and (2) \( u \) must only be adjacent to the vertices \( a \in X_2, b \in X_3, c \in X_4 \).

Hence \( \{v, y_1, a, b, c\} \subset N(u) \), \( |N(u)| \geq 5 \). A contradiction with \( |N(u)| = 4 \).

Finally, two vertices in \( X_i \) are non-adjacent. (2)

+) We consider vertex \( z_1 \):

By (1) and (2), we have \( N(z_1) = \{y_1, a, b, c\} \) where \( a \in X_2, b \in X_3, c \in X_4 \). Without loss of generality, we consider the case \( a = z_4, b = z_5, c = z_{10} \) (if not we can renumber the elements in \( X_i \), \( i = 2, 3, 4 \)).

Hence \( N(z_1) = \{y_1, z_4, z_7, z_{10}\} \).

+) We consider vertex \( z_2 \):

\( N(z_2) = \{y_1, a, b, c\} \) where \( a \in X_2, b \in X_3, c \in X_4 \). We obtain \( a \neq z_4 \) because if \( a = z_4 \), then \( z_4 \) is adjacent to two vertices \( z_1, z_2 \) of \( X_1 \) (a contradiction). Therefore \( z_2 \) is adjacent to \( z_5, z_6 \). Without loss of generality, we only consider the case \( a = z_5 \). Similarly, \( b = z_8 \), \( c = z_{11} \).

Finally, \( N(z_2) = \{y_1, z_5, z_8, z_{11}\} \).

+) The same with vertex \( z_3 \), we have \( N(z_3) = \{y_1, z_6, z_9, z_{12}\} \).

+) We consider vertex \( z_4 \): \( N(z_4) = \{y_2, z_1, a, b\} \).
\[ z_4 \text{ is adjacent to } z_1, \text{ i.e. } z_4 \text{ is not adjacent to } z_2, z_3, y_3, y_4, x. \]

\[ N(y_3) = \{x, z_2, z_7, z_9\}, N(y_4) = \{x, z_{10}, z_{11}, z_{12}\}. \]

Hence \[ z_4 \text{ is adjacent to } z_8, z_{12}, z_4 \text{ is adjacent to } z_8, z_{11}. \]

Without loss of generality, we consider the case \( z_4 \) is adjacent to \( z_8, z_{12} \).

So that \( N(z_4) = \{y_2, z_1, z_8, z_{12}\} \).

+ We consider vertex \( z_5: N(z_5) = \{y_2, z_2, a, b\} \).

\( z_5 \) is not adjacent to \( z_1, z_3, y_3, y_4, x \) so that \( z_5 \text{ is adjacent to } z_7, z_{12} \).

(A contradiction, because \( z_4 \) is adjacent to \( z_{12} \))

Therefore \( N(z_5) = \{y_2, z_2, z_9, z_{10}\} \).

+ We consider vertex \( z_6: N(z_6) = \{y_2, z_3, a, b\} \).

\( z_6 \) is not adjacent to \( z_8, z_9 \). Because \( z_8 \) is adjacent to \( z_4, z_9 \) is adjacent to \( z_5, z_6 \) is adjacent to \( z_7 \). Similarly, \( z_6 \) is adjacent to \( z_{11} \).

Consequently, \( N(z_6) = \{y_2, z_2, z_7, z_{11}\} \).

+ We consider vertex \( z_7: N(z_7) = \{y_2, z_1, z_6, a\} \).

Where \( a \in X_4 \). Because \( z_7 \) is adjacent to \( z_2, z_3, z_4, z_5 \) and \( N(z_2) = \{y_1, z_5, z_6, z_{11}\} \), we have \( a = z_{11} \).

So that \( N(z_7) = \{y_1, z_1, z_6, z_{12}\} \).

On other hand, we have \( N(z_5) = \{y_1, z_6, z_9, z_{12}\} \).

Hence \( N(z_5) \cap N(z_7) = \emptyset \), but \( z_5, z_7 \) are non-adjacent. A contradiction with \( G \) have diameter 2.

There doesn't exist 4 - regular, 2 - diameter graph with 17 edges.

2. Prove that \( n_4 \neq 16 \):

Suppose \( n_4 = 16 \), hence there exists a graph \( G = (V, E) \) with 2 - diameter, 4 - regular and \( |V| = 16 \).

Because \( G \) is a 4 - regular graph, by Theorem 2:

\[ V = \bigcup_{i=1}^{4} N(y_i) \cup \{x\} \cup N(x), \]

where \( N(x) = \{y_1, y_2, y_3, y_4\} \) and \( N(y_i) = \{a, b, c\} \).

If \( y_j \in \{a, b, c\} \) (where \( i \neq j \)), prove that similarly, we have \( |V| \leq 15 \) (A contradiction with \( |V| = 16 \)).

Consequently, \( y_i \) is not adjacent to \( y_j \) (where \( i \neq j \)).

Let \( X_i = N(y_i) - (\{x\} \cup N(x)) \). As above, we have \( |X_i| = 3 \) for all \( i = 1, 4 \).

Let \( \alpha = \max \{|X_i \cap X_j| \colon i \neq j, i, j = 1, 4\} \).

If \( \alpha = 0 \), then \( |X_i \cap X_j| = 0 \) for all \( i \neq j \). Hence, \( X_i \cap X_j = \emptyset \) for all \( i \neq j \).

On other hand, \( V = \bigcup_{i=1}^{4} X_i \cup \{x\} \cup N(x) \). Therefore, we have:

\[ |V| = |\bigcup_{i=1}^{4} X_i| + 1 + 4 = \sum_{i=1}^{4} |X_i| + 5 = 4.3 + 5 = 17. \]

(A contradiction with \( |V| = 16 \)).

So that \( \alpha \geq 1 \). Suppose \( |X_3 \cap X_4| = \alpha \).

\[ 16 = |V| = |\bigcup_{i=1}^{4} X_i| + 5 \leq |X_3 \cup X_2| + |X_3 \cap X_4| + 5 \leq 11 + |X_3 \cup X_4| = 11 + |X_3| + |X_4| - |X_3 \cap X_4| = 17 - \alpha \]

hence \( \alpha \leq 1 \).

Finally, \( \alpha = 1 \) and the equal sign must occur in the ineqation:

So that \( (X_1 \cup X_2) \cap (X_3 \cup X_4) = \emptyset \) and \( X_1 \cap X_2 = \emptyset \).

\[ |X_3 \cap X_4| = 1 \]. Thus, \( X_1 = \{z_1, z_2, z_3\}, X_2 = \{z_4, z_5, z_6\}, X_3 = \{z_7, z_8, z_9\}, X_4 = \{z_9, z_{10}, z_{11}\} \).

![Figure 3. When \( n_4 = 16 \).](image)

+ We consider vertex \( z_9 \): we have \( N(z_9) = \{y_1, y_4, a, b\} \).

Because \( z_9 \) is not adjacent to \( y_1, z_9 \) is adjacent to \( z_1, z_2, z_3 \). Without loss of generality, we consider the case which \( z_9 \) is adjacent to \( z_{14} \).

Similarly, \( z_9 \) is adjacent to \( z_4 \).

Consequently, \( N(z_9) = \{y_3, y_7, z_1, z_4\} \).

+ We consider vertex \( z_2: N(z_2) = \{y_1, a, b, c\} \).

Because \( z_2 \) is not adjacent to \( z_9 \) (where \( N(z_9) = \{y_3, y_4, z_1, z_4\} \)), we obtain \( \{a, b, c\} \cap \{z_1, z_4\} \neq \emptyset \).
Suppose $z_2$ is adjacent to $z_1$, we have $z_2$ is not adjacent to $y_2, y_3, y_4$.

$N(y_2) = \{x, z_4, z_5, z_6\}$, $N(y_3) = \{x, z_7, z_8, z_9\}$,
$N(y_4) = \{x, z_9, z_{10}, z_{11}\}$.

Hence $z_2$ will be adjacent to one of three vertices $z_4, z_5, z_6$, this vertex is called by $a$ , $z_2$ is adjacent to one of three vertices $z_7, z_8, z_9$, this vertex is called by $b$ $(b \neq z_9$ because $z_9 \in N(z_9))$. Similarly, $z_2$ is adjacent to one of two vertices $z_{10}, z_{11}$ and this vertex is called by $c$. Therefore, we obtain $\{y_1, z_1, a, b, c\} \subseteq N(z_2) \Rightarrow |N(z_2)| \geq 5$.

(A contradiction with $|N(z_2)|=4$)

We have $z_2$ is not adjacent to $z_1$, hence $z_2$ is adjacent to $z_4$.

Because $z_2$ is not adjacent to $y_3$, $z_2$ is adjacent to one of three vertices $z_7, z_8, z_9$. Therefore, it is adjacent to one of two vertices $z_7, z_8$. Without loss of generality, we consider the case which $z_2$ is adjacent to $z_7$. Similarly, $z_4$ is adjacent to $z_{10}$. Consequently, $N(z_2) = \{y_1, z_4, z_7, z_{10}\}$.

+ We consider vertex $z_3$:

Similarly as $z_2$, $z_3$ is adjacent to $z_4$ and $z_3$ is adjacent to one of two vertices $z_7, z_8$.

Case $z_3$ is adjacent to $z_7$:

Because $z_2$ is not adjacent to $y_4$, $z_2$ is adjacent to one of two vertices $z_{10}, z_{11}$, this vertex is called by $a$.

$z_7$ is adjacent to one of three vertices $z_4, z_5, z_6$, called by $b$.

$\Rightarrow \{y_2, z_3, a, b, z_2\} \subseteq N(z_7) \Rightarrow |N(z_7)| \geq 5$.

(A contradiction with $|N(z_7)|=4$)

Consequently, $z_3$ must be adjacent to $z_8$. Similarly, $z_3$ is adjacent to $z_{11}$.

Finally, $N(z_3) = \{y_1, z_4, z_8, z_{11}\}$.

+ We consider vertex $z_5$:

Similarly as $z_2$, we obtain $z_5$ is adjacent to $z_1$ and $z_6$ is adjacent to $z_5$.

Because $z_5$ is not adjacent to $y_3$, $z_5$ is adjacent to one of two vertices $z_7, z_8$.

Case $z_5$ is adjacent to $z_7$:

$z_5$ is adjacent to $z_{11}$ (Because $z_5$ is not adjacent to $z_3, y_4$)

Case $z_5$ is adjacent to $z_8$:

$z_5$ is adjacent to $z_{10}$ (Because $z_5$ is not adjacent to $z_2, y_4$)

Without loss of generality, we consider the case which $z_5$ is adjacent to $z_7, z_{11}$

Therefore $z_6$ is adjacent to $z_8, z_{10}$.

Consequently, $N(z_6) = \{y_2, z_1, z_8, z_{10}\}$ and $N(z_6) \cap N(z_7) = \emptyset$.

We have $N(z_7) = \{y_3, z_2, z_4, a\}$.

Because $z_7$ is not adjacent to $y_4$,

$N(y_4) = \{x, z_9, z_{10}, z_{11}\} \Rightarrow |N(z_2)| \geq 5$.

Finally, $n_4 \leq 15$ and $n_4 = 15$. We have $P(3)$ with $|V|=15$, and $P(3)$ is 4-regular graph with diameter 2.

3. Conclusion

In this paper, we define $n_k$ which is maximum number of vertices in a $k$ - regular graph with diameter 2, and we estimate $n_k$ for all $k \in \mathbb{N}$ * and determine $n_k$ in case $k = 2, 3, 4$.

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References

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