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REGULARIZED MULTIQUADRIC METHOD FOR SOLVING INVERSE BOUNDARY VALUE PROBLEMS

Kang Xiao, College of Mathematics, Huaibei Normal University, No.100 Dongshan Road, Huaibei, Anhui 235000, China

Abstract

In this paper, we develop a regularized multiquadric method, which is also a non-iterative numerical method, for solving inverse boundary value problems governed by Laplace equation. The well-known ill-posed Cauchy problem is considered, we assume that the boundary conditions are given only on part of the physical boundary of the solution domain, we have to reconstruct the solution and its normal derivative on the rest un-accessible part of the physical boundary. During the whole solution process, we use the multiquadric and the regularization method to construct a regularized multiquadric method. Numerical experiments are given to demonstrate the effectiveness and efficiency of the proposed method.

Keywords: Gaussian RBF, Nonlinear PDEs, Variable coefficient PDEs.

1 INTRODUCTION

Scattered data approximation has been a fast growing research area. It deals with the problem of reconstruction of an unknown function from given scattered data [1]. Naturally, it has many applications, such as surface reconstruction, fluid-structure interaction, the numerical solution of partial differential equations and parameter estimation [2]. Moreover, these applications come from such different fields as applied mathematics, computer science, biology, and engineering. Increasingly parabolic inverse problems play a crucial role in applied mathematics and physics. Thus investigation of these kinds of partial differential equations has been recently addressed by various authors. A growing attention is seen in the literature to the development, analysis, and copy with effective methods for the numerical solution of parabolic inverse problems with specified boundary data in these 15 years [3]. And many scholars have studied the global collocation method for the various fields. Even the location of boundary itself is sometimes unknown resulting from special geological features. Therefore, the inverse boundary value problem (IBVP) arises to deal with the estimation of the recovered boundary conditions on unspecified part of the boundary. In order to make the problem solvable, partial accessible boundary may be over-specified for example, the Dirichlet and the Neumann conditions are simultaneously prescribed, and in some cases the auxiliary internal data are necessary for input. The IBVPs arise from many engineering applications such as heat transfer [4], geophysical prospecting [5], medical imaging and non-destructive testing [6] and acoustic and electromagnetic waves [7]. It is well known that the inverse problems have a typical characteristic of ill-posedness in the sense that a slight error in the input data may produce an enormous change in the output solution that makes them more difficult to deal with. However, the measurement inevitably poses some noises due to the technical and physical difficulties. It is required to develop a numerical method to solve the inverse problem with noisy data input. Whichever mesh less method is applied, the inverse problems can be converted into a large-scale system of linear algebraic equations in which a number of unknown coefficients need to be considerable. However, the coefficient matrix is inherently ill-conditioned and the solution is highly sensitive to the noise of measured input data. Many regularization methods are additionally employed to obtain stable solutions, for example, the standard TR technique with the L-curve criterion (LC) [8, 9], the truncated singular value decomposition (SVD) with the LC [10] and three regularization strategies TR, truncated SVD and damped SVD under the different choices for the regularization parameter [11]. Mao and Li [12] employed the least-square technique using the SVD to restore the stability with large noise level. According to the results of literature survey, mesh less local collocation method based on the multiquadric and inverse multiquadric radial basis functions has been applied in many fields, and the traditional RBFs are globally defined functions which result in a full resultant coefficient matrix. This hinders the application of the RBFs to solve large-scale problems due to severe ill-conditioning of the coefficient matrix. To tackle this we approximately using radial basis function, the shape parameter has the important effect on the approximation accuracy. In this paper, radial basis function methods with its simple implementation as a mesh-free method is examined. We develop a regularized multiquadric method, which is also a non-iterative numerical method, for solving inverse boundary value problems governed by Laplace equation. We consider two examples, in the first example, we consider a We take an exact solution which domain with the Dirichlet and Neumann data given on an inner circle, and solved by a regular technique using MQ. In Example 2, we consider a 2D inhomogeneous Helmholtz equation case, which with the Dirichlet and Neumann data and wave number. We use the MQ method could achieve the error figure via the software MATLAB.

2 FORMULATION OF MQ

For simplicity, let $\Omega \in R^d$ we consider the following boundary value problem

$$Lu = f \quad \text{in } \Omega \qquad (1)$$

$$u = g \quad \text{on } \partial\Omega \qquad (2)$$

Where L represents a linear differential operator, d is the dimension of problem. As for nonlinear operators, some kinds of linearization ways will be needed to seek the solution iteratively. For numerical verification opinions, we will focus on solving the following

Poisson equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \qquad (x, y) \in \Omega, \tag{3}$$

and Helmholtz-type equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda^2 u = f(x, y), \qquad (x, y) \in \Omega, \tag{4}$$

With λ the wave number.

Let $\{(x_j, y_j)\}_{i=1}^{N}$ be N distinct collocation points in Ω of which $\{(x_j, y_j)\}_{i=1}^{N1}$ interior points and $\{(x_j, y_j)\}_{i=N1+1}^{N}$ are boundary points. Due to the exponential convergence and boundary points. Due to the exponential convergence and superior performance of MQ we indicated in section A, MQ is the one of the most widely adopted RBFs in Kansa's method. Though other RBFs can be used, we consider only Hard's RBF-MQ basis function.

$$\varphi_j(x, y) = \sqrt{(x - x_j)^2 + (y - y_j)^2 + c^2} = \sqrt{r_j^2 + c^2}, \tag{5}$$

$$r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2} \tag{6}$$

from which we have

$$\frac{\partial \varphi_j}{\partial x} = (x - x_j)/\sqrt{r_j^2 + c^2}, \tag{7}$$

$$\frac{\partial \varphi_j}{\partial y} = (y - y_j)/\sqrt{r_j^2 + c^2} \tag{8}$$

$$\frac{\partial^2 \varphi_j}{\partial x^2} = (y - y_j)^2 + \frac{c^2}{(r_j^2 + c^2)^{3/2}}, \tag{9}$$

$$\frac{\partial^2 \varphi_j}{\partial y^2} = (x - x_j)^2 + \frac{c^2}{(r_i^2 + c^2)^{3/2}} \tag{10}$$

Where c is the shape parameter of MQ.

For the elliptic problem (3) and (4), the idea of Kansa's method is to approximate the solution u by

$$u(x, y) = \sum_{j=1}^{N} c_j \varphi_j(x, y), \tag{11}$$

Where c_j are coefficients to be determined. Next, consider the Cauchy problem for linear elliptic equations of second order:

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \Omega \subseteq R^2 \tag{12}$$

$$u = g(x, y) \text{ and } \frac{\partial u}{\partial n} = h(x, y) \qquad \Gamma 1 \tag{13}$$

To solve our inverse problem (12)-(13), we assume that the approximate solution can be expressed as

$$u_N(x, y) = \sum_{j=1}^{N_I + N_d + N_n} u_j \varphi(\|(x, y) - (x_j, y_j)\|) \tag{14}$$

Where u_j are the unknown coefficients to be determined, and $\varphi(r_j)$ is some kind RBF. Here $r_j = \|(x, y) - (x_j, y_j)\|$ is the Euclidean norm between points $P(x, y)$ and $Pj\ (xj, yj)$. We denote$\{(x_j, y_j)\}_{I}^{NI}$ the collocation points inside the domain Ω, while$\{(x_j, y_j)\}_{NI+1}^{NI+Nd}$, $\{(x_j, y_j)\}_{NI+Nd+1}^{NI+Nd+Nn}$ are the collocation points on the boundary $\Gamma 1$ for Diichlet and Neumann conditions, respectively. Although both Dirichlet and Neumann conditions are given on the same boundary$\Gamma 1$, we use different nodes for imposing each type boundary condition. By forcing (14) to satisfy (12)-(13) at the collocation points, we have

$$\sum_{j=1}^{N} L\varphi(\|(x_i, y_i) - (x_j, y_j)\|)u_j = f(x_i, y_i), \ i = 1, 2, \dots, N_l \tag{15}$$

$$\sum_{j=1}^{N} \varphi(\|(x_i, y_i) - (x_j, y_j)\|)u_j = g(x_i, y_i), \ i = N_l + 1, \dots, N_l + N_d \tag{16}$$

$$\sum_{j=1}^{N} (\frac{\partial \varphi}{\partial n}(\|(x_i, y_i) - (x_j, y_j)\|)u_j = h(x_i, y_i), \ i = N_l + N_d + 1, \dots, N \tag{17}$$

Which can be solved for the unknown coefficients. Here $N = N_I + N_d + N_n$ is the total number of collocation points. Equations (15)-(17) can be written in the following matrix system:

$$A\alpha_j = b. \tag{18}$$

An interesting and significant aspect of discrete ill-posed problems is that the ill-conditioning of a given problem does not prevent us from getting meaningful approximate solutions. Rather, it implies that the standard methods in numerical linear algebra for solving Equation such as Gaussian elimination, may not be suitable for solving this type of problems. As such, regularization methods are proposed to alleviate the difficulty of highly ill-conditioning problems. We briefly introduce some of them in the following section.

3 REGULATION METHODS

Before presenting our numerical results, we give a brief discussion of some regularization methods.

3.1 Singular value decomposition (SVD)

As is well known, the matrix An in Equation (14) can be decomposed as [13],

$$A = UDV^T \tag{19}$$

Where $U = [u1, u2\dots u_N]$ and $V = [v1, v2, .. v_N]$ are matrices with orthogonal columns, $U^T U = V^T V = I_N$ the superscript T represents the transpose of a matrix, IN denotes the identity matrix, and D is a diagonal matrix with diagonal elements,

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_N \geq 0 \tag{20}$$

and right singular vectors of *A*, respectively. Using Equation (15), we can solve Equation (14) in the following form:

$$\alpha = \sum_{i=1}^{N} (u_i^T) b v_i / \sigma_u \quad (21)$$

We note that the ill-conditioning of A is due to the small singular values as shown in the denominator of (23). Based on the SVD, we present some commonly used regularization methods for ill-posed problems in the following subsection.

3.2 TR method

One of the most popular regularization methods is the TR, which in its simplest form replaces the linear system (20) by the minimization problem

$$\min_{\alpha \in R^n} \{\|A\alpha - b\|^2 + u^2\|\alpha\|^2\} \quad (22)$$

Here $U \geq 0$ is a regularization parameter. The Tikhonov-regularized technique based on SVD can then be express as

$$\alpha_u = \alpha_m in = \sum_{i=1}^{l} (u_i^T) f_i b v_i / \sigma_u \quad (23)$$

Where the Wiener weights are

$$f_i = (\sigma_i)^2 / (\sigma_i^2 + u^2) \quad (24)$$

and *l* is the rank of *A*.

3.3 Regularization parameters

The determination of a satisfactory value for the regularization parameter *u* is crucial and is still under intensive research. In this paper, we use the LC criterion and the GCV to choose a good regularization parameter. *LC* for choosing the regularization parameter [14, 15]: A proper choice of the regularization parameter *u* is essential in the successful use of a regularization method. Define a curve

$$L; = [(\log\|\alpha_u\|, \log\|A\alpha_u - b\|) : u \geq 0] \quad (25)$$

The above curve is referred to as the L-curve, because it is shaped like the letter L for a large class of problems. We note here that the L-curve is a continuous curve when the regularization parameter is real in the TR and the DSVD. In numerical computation, the point with maximum curvature will be searched as the corner of the L-curve. For the regularization methods with a discrete regularization parameter, such as in TSVD, a finite set of points

$$[(\log\|\alpha_q\|, \log\|A\alpha_q - b\|) : q = 1, 2, ...N] \quad (26)$$

Will be obtained and interpolated by a spline curve. The L-curve is very attractive because the method shows how the regularized solution changes with the regularization parameter *u*.

4 NUMERICAL EXAMPLE

To examine the accuracy and stability of the proposed regularization methods given in the above sections, we test two cases of homogeneous Helmholtz and modified Helmholtz problems. The relative average error (root mean-square relative error: RMSE) is used. The convergence behavior of the BKM using three regularized methods are shown in the given curves of the relative average error versus the number of boundary knots. MATLAB regularization code developed by Hansen [14] has been used in our computations. In this section several numerical experiments are performed to show the efficiency and robustness of the proposed method. Past experiments show that the accuracy of the Kansas method is very sensitive to the choice of parameters in RBFs (e.g. the multiquadric and Gaussians, the support scaling factor in compactly supported RBFs), we simply choose $\varphi(r) = r^{11}$ in the all our computations due to its parameter-free property and high accuracy compared to lower-order basis functions.

4.1 Example 1

We take the exact solution *u*(*x; y*) as

$$u(x, y) = -0.5x + 2y + 0.25(x^2 - y^2) \quad (27)$$

and the domain $\Omega = \{(x, y) | 1 \leq (x^2 + y^2)^{\frac{1}{2}} \leq 3\}$ with the Dirichlet and Neumann data g(x, y) =u, h(x, y) = $\partial u/\partial n$ given on the inner circle$\tau 1$.

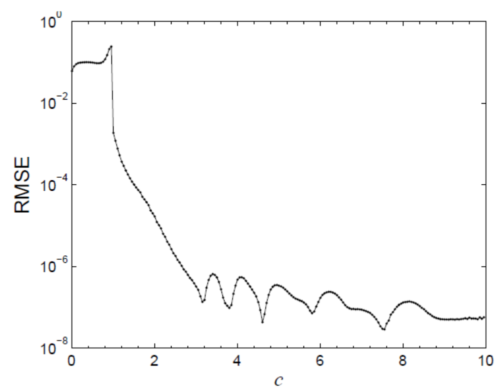

Fig. 1: influence of the shape parameter *c* on RMSE for the case *n* = 15.

$x^2 + y^2 = 1$. This example was consider by [5], which is solved by a regular technique using MQ. On the one side, we control the collocation point number parameter *n*, observe the change of the shape parameter *c*, from 0.01 to 10, with the interval 0.05. We can clear find the phenomenon that the *c* reach to 3, the error will be very small, and the fluctuation still exist when the figure over 3, while the error stay in the small range with the all data. AS a result, we select the optimal *c* = 7:56. On the other side, we make the shape parameter *c* at 7.56, transfer the collocation point number parameter *n*. Then we obtain the figure which exhibit the phenomenon *c*=7.56 the error has been small. We use the method is good at decrease the error make the model precise.

4.2 Example 2

Next we consider a 2D inhomogeneous Helmholtz equation case given by we

$$\nabla^2 u + \lambda^2 u = 2\sin(\mu x)\cos(\mu y) + 4\mu x\cos(\mu x)\cos(\mu y) \tag{28}$$

$$u = x^2 \sin(\mu x)\cos(\mu y) \tag{29}$$

with the Dirichlet and Neumann data $g(x; y) = u$; $h(x; y) = \partial u/\partial n$. With wave number($\lambda = \sqrt{2}$), we are able to achieve the error figure in following.

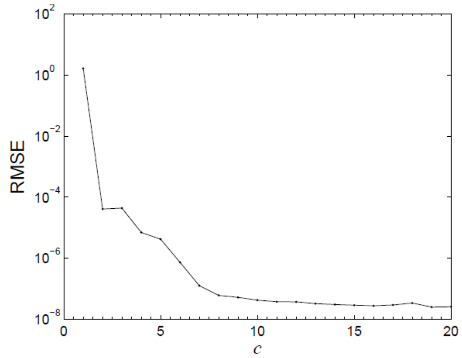

Fig. 2: Influence of the collocation point number parameter n on RMSE for the case $c = 7{:}56$.

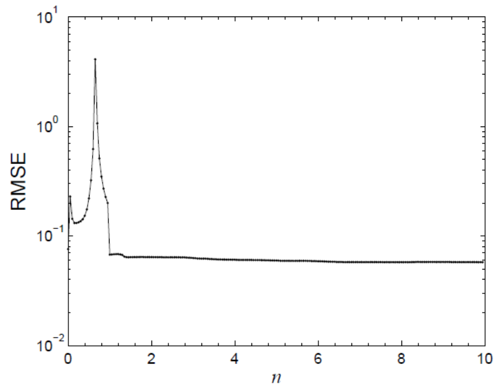

Fig. 3: influence of the shape parameter c on RMSE for the case n = 15.

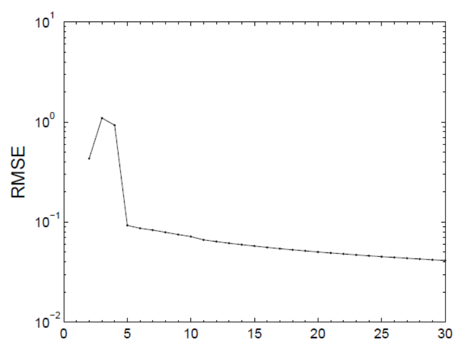

Fig. 4: Influence of the collocation point number parameter n on RMSE for the case $c = 7{:}56$

Figure 3 shows the influence of the shape parameter c on RMSE. When the $n = 15$, the error has been control in a small range. It's clear to see if n over 1.5, the error is drop to a minimums and in the later of increase the n's figure the error nearly become still. When we seek the regulation of influence of the collocation point number parameter n on RMSE for the case $c = 7{:}56$, we can realize n over 5, the error can be fall in a relative accuracy range.

5 CONCLUDING REMARKS

In this paper, a global MQ method is proposed as a spatial-temporal approximation to solve the inverse boundary value problems governed by Laplace equation. The new method incorporates time dimension into the MQ function as a new variable in radial coordinate in the entire space C time domain. As a result, the time-dependent ill-posed problem can be solved as a direct problem. This over determined linear system with the use of two sets of collocation points: one is satisfied with the governing equation and the other is for the given conditions. To overcome the ill-conditioned resultant matrix, the least-square technique is introduced to find the best-¯t solution of the over determined linear system. And the capability of the LR-MQ to resist the noise is a potentially powerful tool for practical inverse problems in engineering applications.

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